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Robert E. Horton, Editor   Donald H. Hyers, Associate Editor
Vance D. Stine, Circulation Manager

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THE EDITOR'S PAGE

Au Revoir

If the reader will turn to the title page of this issue, he will find listed there the names of the editors and staff of the *Mathematics Magazine*. These are the men who have set the policies, selected the manuscripts, and performed the details required to publish the magazine. Many of them have served the magazine for ten or more years.

As has been announced previously, the *Mathematics Magazine* has become an official publication of the Mathematical Association of America. A new editor and staff will be appointed by the Governors of the Association. These editors will be responsible for publishing the issues of the magazine subsequent to this current issue. The composition of the new editorial staff will be announced in the September, 1961 issue.

A number of expressions of appreciation have been received for the loyal and effective service rendered by the retiring editors. Their service over the years has made the magazine a valuable organ in the field of collegiate mathematics. I am sure the readers of the *Mathematics Magazine* and the members of the mathematical profession join me in adding our word of thanks to all the editors and staff whose long and unselfish service to the magazine is ending now.

The Long Count

At the Forty-fourth Annual Meeting of the Association, the Board of Governors directed that the volume year of the *Mathematics Magazine* be changed to coincide with the calendar year. To accomplish this, the current volume, Volume 34, will be extended to the end of 1961, and Volume 35 will begin with the January, 1962 issue.

Librarians and others concerned should note that, as a result of the change, Volume 34 will contain seven issues instead of the usual five.

R. E. H.

SATELLITE MECHANICS

Verner E. Hoggatt

This paper is written with only one thing in mind. Its primary purpose is to furnish a little historical background of the solar system and the mathematical-physical inductive-deductive system of logic. Secondarily the results of Kepler and Newton are shown compatible through the use of differential equations and the physical and mathematical notions of kinetic and potential energy.

The situation is an idealized one which can be used to get very rough estimates of actual satellite behavior in collision or non-collision orbits. From initial conditions such as the velocity and altitude vector at which free flight takes place, one can get an idealized orbit.

Since earliest times man has desired to predict the positions of the stars and planets in their endless wandering in space.

The Ptolemiac Theory of the solar system was expounded by Ptolemy (150 A.D.) and was geocentric (earth-centered). The Copernican Theory of the solar system was heliocentric (sun-centered), and involved the planets traveling in concentric circles about the sun. Nicolaus Copernicus (1473-1543), a Polish astronomer, devised it to overhaul the Ptolemiac Theory. The widespread acceptance of the Copernican theory was brought about by: (1) the invention of the telescope by Johann Lippersheim of Holland 1607, (2) Galileo's observation of the moons of Jupiter which are Jupiter s satellites (1633).

Johann Kepler was born in 1571 and in 1599 was made assistant to the quarrelsome Danish-Swedish astronomer Tycho Brahe. Tycho Brahe was in possession of a set of very accurate astronomical observations made by himself.

Tycho Brahe became Court Astronomer to Kaiser Rudolph II in 1601 but died suddenly and Kepler inherited both the high job and the collection of accurate data.

Kepler in his youth had devised a correspondence between the planets (radius of circles around the sun) and the polyhedra (radius of inscribed and exscribed spheres) which he called "Mysterium Cosmographicum". Even later to this physical-geometric picture he added sound in his "De Harmonici Mundi" in which he talks of the *music* of the spheres heard only by God. (*The Bequest of the Greeks* by Tobias Dantzig). *Note : De Broglie in modern times uses the same notion but now we can detect it.

While these mysterious things were going on in the mind of Kepler he also started on a gigantic task of inductive logic, that of obtaining from the astronomical data a design of the solar system. It is estimated he spent some 20 years of hard trial and error computation before he evolved his empirical three laws of planetary motion.

These laws of planetary motion are landmarks in the history of astronomy and mathematics.

I. The planets move about the sun in elliptical orbits with the sun at one focus.

II. The radius vector joining a planet to the sun sweeps out equal areas in equal times.

III. The square of the time of one revolution of a planet about its orbit is proportional to the cube of the orbit's semimajor axis. (*History of Mathematics*, Howard Eves).

Sir Isaac Newton was born the year Galileo died. (1642) He espoused the three laws of Newtonian Mechanics as follows:

I. Every body will continue in its state of rest or of uniform (unaccelerated) motion in a straight line except in so far as it is compelled to change that state by an impressed force.

II. The rate of change of momentum (mass × velocity) is proportional to the impressed force and takes place in the line in which the force acts.

III. Action and reaction (as in the collision on a frictionless table of two perfectly elastic billiard balls) are equal and opposite. (The momentum one ball loses is gained by the other.) (*Men of Mathematics* by E.T. Bell)

Newton used Kepler's laws inductively.

Complex Numbers

A particle moving along a constrained path is subject to certain body forces due to this motion. It is convenient to find out what these must be in order to study the satellite problem. The following method was suggested in reference 3, page 193, problem 45.

Let $z(t) = x(t) + iy(t)$ be the parametric form of the complex number as conceived to be plotted on the Gauss plane. In reality this is a convenient way of handling two different parametric equations at once. As t (time) varies the tip of the line segment from $(0, 0)$ to (x, y) traces out a curve in the (x, y) plane. If we assign the unit direction vectors to be 1 (in $+x$ direction) and $i = \sqrt{-1}$ (in $+y$ direction), then $z(t)$ becomes (in analog) a directed line segment from $(0, 0)$ to (x, y).

Let $z(t)$ be a position vector at each instant of time. A particle moving along a plane curve can be described by $z(t)$ when $x = x(t)$ and $y = y(t)$ is a parametric representation of the plane curve. But we gain more than compactness.

Next $\dot{z}(t)$ is a velocity vector along the tangent line to the curve of motion and is in the direction of motion and has the usual horizontal and vertical resolution into components $\dot{x}(t)$ and $\dot{y}(t)$. Similarly $\ddot{z}(t)$ is an acceleration vector resolved into components $\ddot{x}(t)$ and $\ddot{y}(t)$.

However complex numbers can be written in several ways:

$$z(t) = \frac{x + iy}{\text{rectangular}} = \frac{r(\cos\theta + i\sin\theta)}{\text{polar or trigonometric}} = \frac{re^{i\theta}}{\text{exponential}}.$$

We first note that $e^{i\theta}$ is a unit complex number whose tip traverses the unit circle as θ is increased by 2π radians. It is easy to see from Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ that $e^{i\pi/2} = i$. Next $e^{i\pi/2}e^{i\theta} = e^{i(\theta+\pi/2)} = ie^{i\theta}$ so that $e^{i\theta}$ and $ie^{i\theta}$ are at right angles. This can be easily deduced also from the fact that their slopes are negative reciprocals.

We now wish to resolve position, velocity, and acceleration vectors into components along the unit vectors $e^{i\theta}$ (along z) and $ie^{i\theta}$ (perpendicular to z).

(A) $$z(t) = r(t)e^{i\theta(t)} \text{ (position vector)} .$$

(B) $$\frac{dz}{dt} = \dot{z}(t) = \dot{r}(t)e^{i\theta(t)} + re^{i\theta(t)}[i\dot{\theta}(t)]$$
$$= \dot{r}[e^{i\theta}] + r\dot{\theta}[ie^{i\theta}] = v_r + v_\theta \text{ (velocity vectors)} .$$

The component of velocity along the direction of z is $\dot{r} = \dfrac{dr}{dt}$ and the component of velocity perpendicular to z is $r\dot{\theta} = r\dfrac{d\theta}{dt} = rw$.

Differentiating Eq. B, with respect to time, we obtain

$$\frac{d^2z}{dt^2} = \ddot{z} = \underbrace{\ddot{r}e^{i\theta} + \dot{r}e^{i\theta}(i\dot{\theta})}_{\frac{d}{dt}(\dot{r}e^{i\theta})} + \underbrace{re^{i\theta}(i\ddot{\theta}) + re^{i\theta}(i\dot{\theta}i\dot{\theta}) + \dot{r}e^{i\theta}(i\dot{\theta})}_{\frac{d}{dt}(re^{i\theta}i\dot{\theta})} .$$

Regrouping we obtain:

$$\ddot{z} = [e^{i\theta}][\ddot{r} - r\dot{\theta}^2] + [ie^{i\theta}][2\dot{r}\dot{\theta} + r\ddot{\theta}] .$$

Therefore $a_r = \ddot{r} - r\dot{\theta}^2$ is the acceleration component, along z, called the radial acceleration and $a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$ is the acceleration component perpendicular to z, called the angular acceleration.

The Satellite Problem

The satellite is in free flight in a vacuum subject to an impressed force of gravitational attraction from a spherical fixed earth of mass M and the ensuing motion is planar being determined by the starting velocity vector and the line of centers of the earth and satellite.

Newton's law of gravitational attraction states $F = -\gamma Mm/r^2$. Therefore

(1) $$F_r = -\frac{\gamma Mm}{r^2} = m(\ddot{r} - r\dot{\theta}^2) = \frac{d}{dt}(mv) ,$$

where m is the assumed constant mass, and

(2) $$F_\theta = 0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) .$$

These are the two differential equations in three variables (r, θ and t) from which we must obtain a solution.

Multiplying (2) through by an integrating factor r/m we see that

$$\frac{d}{dt}[r^2\dot{\theta}] = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0 ,$$

whence

(3) $$r^2\dot{\theta} = h \quad \text{(constant)} .$$

Recalling that $\frac{1}{2}r^2 d\theta$ is the differential of area in polar coordinates, then $\frac{1}{2}r^2\frac{d\theta}{dt}$ is the rate of generation of area, $A(t)$, swept out by the radius vector. Since $\frac{dA}{dt} = \frac{h}{2}$ (a constant), *Kepler's Second Law* is found to be true under the inverse square hypothesis.

We note in solving (2) we did not use (1) the universal law of gravitation (inverse square), so that *Kepler's Second Law* remains true under more general gravitational laws.

To solve (1) we'll need the solution to (2) which is (3) and we wish to eliminate the time variable, t. Let $u = \frac{1}{r}$, then from (3), $r^2\frac{d\theta}{dt} = h$ and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta} ,$$

since

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} .$$

Also

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} ,$$

since

$$-h\frac{d\theta}{dt} = (-h)\left(\frac{h}{r^2}\right) = -h^2u^2$$

from (3). Equation (1) now becomes

$$-mh^2u^2\frac{d^2u}{d\theta^2} - mh^2u^3 = -\gamma Mmu^2 ,$$

an equation, free of t, in the variables u and θ. Dividing through by $-mh^2u^2 \neq 0$ this becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma M}{h^2} .$$

This type of problem is studied in Chap. VI of Kell's differential equation using the D-operator. It's solution is

$$u = u_c + u_p = \underbrace{A'\sin\theta + B'\cos\theta}_{u_c} + \underbrace{(\gamma M/h^2)}_{u_p} .$$

This can be put in the form

$$u = \frac{\gamma M}{h^2} + A\cos(\theta+\delta)$$

by letting

$$\cos\delta = \frac{B'}{\sqrt{(A')^2+(B')^2}}, \quad \sin\delta = -\frac{A'}{\sqrt{(A')^2+(B')^2}}, \quad \text{and } A = \sqrt{(A')^2+(B')^2}\,.$$

Thus A and δ are the two arbitrary constants required for the general solution.

Replacing u by $1/r$ and recalling the polar form of a conic section with pole at a focus which is

$$r = \frac{pe}{1+e\cos\theta},$$

where p is distance from a focus to the corresponding directrix and e is the eccentricity.

Equation (4) now becomes

$$r = \frac{h^2/\gamma M}{1+(Ah^2/\gamma M)\cos(\theta+\delta)}\,.$$

Assuming now that $|e| = |Ah^2/\gamma M| < 1$, and $A > 0$, then, for $\theta+\delta = 0$, r will have a minimum, and, for $\theta+\delta = +\pi$, r will have a maximum.

Thus $\delta = \pi$ or 0 corresponds to putting the polar axis along the major-axis of the ellipse with the pole at one focus. We choose $\delta = \pi$.*

In case $|e| \geqq 1$, the satellite will not orbit but has escaped since it is on a parabola or one branch of an hyperbola. So far none have escaped. For $|e| < 1$ we have *Kepler's First Law.*

We next show that the total energy of the satellite is a constant once it is in free flight. The potential energy of a satellite can be defined.

$$\text{P.E.} = V = \int_{\infty}^{r} + \frac{\gamma Mm}{r^2}\,dr = -\frac{\gamma Mm}{r},$$

where $V(r\to\infty) = 0$. The kinetic energy is usually defined as $\frac{1}{2}mv^2$. Since $v = ds/dt$, we note in polar form that

$$\text{K.E.} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \tfrac{1}{2}m(v_r^2 + v_\theta^2)\,.$$

Both (1) and (2) are force equations:

(1) $$m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = -\gamma\frac{Mm}{r^2},$$

---

*Note: This will give a positive eccentricity, $e = Ah^2/\gamma M$, consistent with $A = \sqrt{(A')^2+(B')^2} > 0$, since h^2 and γM are positive. Therefore $r = pe/(1-e\cos\theta)$.

(2) $$m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = 0 \ .$$

If these are multiplied through by a distance and integrated, we will get energy or work. We do this in three steps.

Multiply (1) by $\frac{dr}{dt}$ and (2) by $r\frac{d\theta}{dt}$, both velocity terms, and adding

$$m\left(\frac{dr}{dt}\frac{d^2r}{dt^2} + \left[-r\left(\frac{d\theta}{dt}\right)^2\frac{dr}{dt}\right] + 2r\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)^2 + r^2\frac{d^2\theta}{dt^2}\frac{d\theta}{dt}\right) = -\frac{\gamma Mm}{r^2}\frac{dr}{dt} \ .$$

Combining

$$m\frac{dr}{dt}\frac{d^2r}{dt^2} + r\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)^2 + r^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -\frac{\gamma Mm}{r^2}\frac{dr}{dt} \ .$$

But

$$\frac{d}{dt}\left\{\tfrac{1}{2}m\ \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} = \frac{m}{2}\left\{2\frac{dr}{dt}\frac{d^2r}{dt^2} + 2r\left(\frac{d\theta}{dt}\right)^2 + r^2\left(2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right)\right\}$$

$$= -\frac{d}{dt}\left(\frac{\gamma Mm}{r}\right) = -\frac{dV}{dt} \ .$$

After multiplying through by dt* and transposing, equation (6) becomes

(7) $$d[\text{K. E.} + \text{P. E.}] = d[\tfrac{1}{2}mv^2 + (-\frac{\gamma Mm}{r})] = 0 \ .$$

Equation (7) states that the total energy is a constant. Thus

$$\tfrac{1}{2}mv^2 - \frac{\gamma Mm}{r} = u \text{ (a total energy, a constant) .}$$

We may note if $v \to 0$ as $r \to \infty$, then $u = 0$.

We next relate u, the total energy, to the major axis of the ellipse of motion. Suppose $A > 0$,

$$r = \frac{h^2/\gamma M}{1 - (Ah^2/\gamma M)\cos\theta} \quad \text{or} \quad \frac{pe}{1 - e\cos\theta}$$

and

$$r_{min} = \frac{h^2/\gamma M}{1 + (Ah^2/\gamma M)} \quad \text{and} \quad r_{max} = \frac{h^2/\gamma M}{1 - (Ah^2/\gamma M)},$$

$$2a = r_{min} + r_{max} = (h^2/\gamma M)\left\{\frac{2}{1 - (Ah^2/\gamma M)^2}\right\}$$

or

(9) $$\frac{1}{a} = \frac{\gamma M}{h^2}\left\{1 - \left(\frac{Ah^2}{\gamma M}\right)^2\right\} = \frac{\gamma M}{h^2}\{(1 - e^2)\} \ .$$

---

* $(\frac{dr}{dt})dt$ is distance and $r\frac{d\theta}{dt}dt$ is distance.

To evaluate u at a particular point in the orbit (whence we know it for all points), we note that at the perigee (closest approach to the pole which is the focus) that necessarily the motion is all tranverse ($\frac{dr}{dt} = 0$). The velocity at that point is $v = v_\theta = R\frac{d\theta}{dt} = \frac{h}{R}$, where $R = r_{min}$ and $R^2\frac{d\theta}{dt} = h$ was used from equation (3). From solution (4) with $\delta = 0$,

$$\frac{1}{R} = \frac{\gamma M}{h^2} + A \quad \text{so that} \quad v\text{ (at perigee)} = (\frac{\gamma M}{h} + Ah) .$$

Putting these into the total energy equation we have

$$\tfrac{1}{2}mv^2 - \frac{\gamma Mm}{R} = \tfrac{1}{2}m[\frac{\gamma M}{h} + Ah]^2 - Mm[\frac{\gamma M}{h^2} + A] = u .$$

Simplifying

$$\frac{2u}{m} = \frac{\gamma^2M^2}{h^2} + 2\gamma MA + A^2h^2 - 2\frac{\gamma^2M^2}{h^2} - 2\gamma MA$$

$$= \frac{\gamma^2M^2}{h^2} + A^2h^2 = -\frac{\gamma^2M^2}{h^2}(1 - \frac{A^2h^4}{\gamma^2M^2})$$

$$= -\frac{\gamma M}{1}(\frac{\gamma M}{h^2})\{(1-(\frac{Ah^2}{\gamma M})^2\} = -\frac{\gamma M}{a}, \text{ from (9)} .$$

Therefore

$$u = -\frac{\gamma Mm}{2a} . \tag{10}$$

Thus the total energy determines the major axis of the ellipse and conversely. We note that $u = 0$ demands $2a = \infty$.

Therefore for each point of orbit

$$\tfrac{1}{2}mv^2 - \frac{\gamma Mm}{r} = -\frac{\gamma Mm}{2a} . \tag{11}$$

We next turn to the period of revolution. From Eq. (3)

$$\tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}h \quad \text{or} \quad \int_{\theta=0}^{\theta=2\pi}\tfrac{1}{2}r^2 d\theta = \tfrac{1}{2}\int_0^P h\,dt ,$$

where P is a period of revolution for sweeping out the ellipse once. Since the area of an ellipse is πab, then

$$\tfrac{1}{2}hP = \pi ab .$$

The eccentricity of an ellipse is $e = c/a = \sqrt{a^2 - b^2}/a$ so that $b = a\sqrt{1-e^2}$. Therefore, since $(1/a) = (\gamma M/h^2)(1-e^2)$ from (9),

$$P = \frac{2\pi}{h}(a^2\sqrt{1-e^2}) = \frac{2\pi}{h}\left((a^2)\{\frac{h^2}{\gamma M}\frac{1}{a}\}^{1/2}\right)$$

(12) $$P = \frac{2\pi a^{3/2}}{\sqrt{\gamma M}} .$$

This is *Kepler's Third Law.*

To get the particular ellipse given r and v at instant of free flight. From Eq. 5

$$r = \frac{B}{1 - AB\cos\theta}, \quad \text{where} \quad B = \frac{h^2}{\gamma M},$$

then

(13) $$A\cos\theta = \frac{r-B}{Br} .$$

Further,

(14) $$\dot{r} = -\frac{AB^2\sin\theta}{(1 - AB\cos\theta)^2}\dot{\theta} = -A\sin\theta(r^2\dot{\theta}) .$$

But $h = r^2\dot{\theta}$, also $h = rv\cos\phi$ and $\dot{r} = v\sin\phi$, whence $A\sin\theta = -\frac{\dot{v}}{r^2\dot{\theta}} = -\frac{\dot{r}}{h}$ and $-\frac{v\sin\phi}{rv\cos\phi} = -\frac{\tan\phi}{r}$, where ϕ is the angle of the velocity vector from the horizontal. Therefore, at the instant of free flight,

$$\tan\theta_0 = +\frac{A\sin\theta_0}{A\cos\theta_0} = -\frac{(1/r)\tan\phi_0}{(r-B)/Br}$$

(15) $$\tan\theta_0 = \frac{B}{B-r}\tan\phi_0 .$$

Thus the orientation of the polar axis is easily found once the vector orientation is given. We therefore need $h = rv\cos\phi$, and γM, and radius r. From $\frac{\gamma M}{2a} = \frac{\gamma M}{r} - \frac{1}{2}v^2$, we get the major axis $2a$. From $\frac{Ah^2}{\gamma M} = \frac{c}{a} = e$ and $b^2 = (1-e^2)a^2$, one can get the eccentricity and the semi-minor axis.

Additional Remarks

In order to fire a projectile to escape the earth (drift to infinity) with minimum effort, one would want $v \to 0$ as $r \to \infty$. We have already defined $V \to 0$ as $r \to \infty$. Therefore, if we wish to achieve escape velocity we must have:

$$\tfrac{1}{2}mv^2 - \frac{\gamma Mm}{r} = 0 \quad \text{or} \quad v = \sqrt{\frac{2\gamma M}{r}} ,$$

Thus the larger the r at which free flight takes place the smaller the velocity your bird has to have to escape.

Since the total energy determines the major axis, the earth the focus, the free-flight velocity in direction and magnitude determines the eccentricity.

If $b < 4000$ miles, the rocket will crash back to earth even in orbit since the ellipse intersects the earth. A near circular orbit is hard to achieve as the velocity (terminal) is critical.

Specifications for Two Ellipses

I. *One datum* could be: $\phi = 36.5°$, $r = 2.1174 \times 10^7$ ft., $v = 10^4$ ft./sec.

II. *Another datum* could be: $\phi = 20°$, $r = 4.1808 \times 10^7$ ft., $v = 2 \times 10^4$ ft./sec.

The first ellipse is a collision orbit but the second ellipse is a non-collision orbit.

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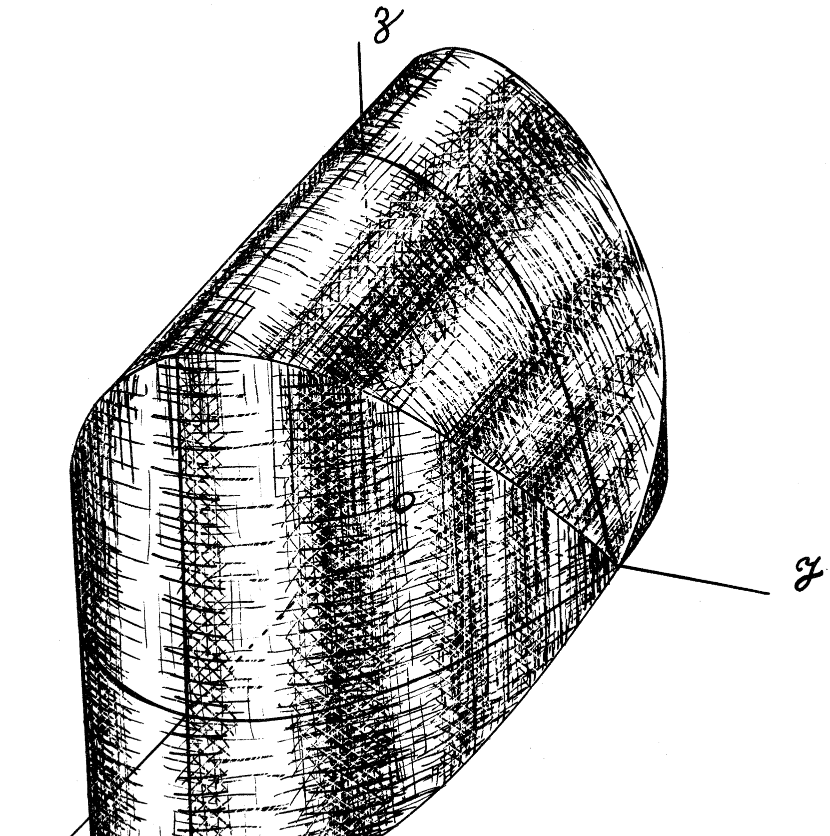
San Jose State College
San Jose, Calif.

Carl Friedrich Gauss. G. Waldo Dunnington, Exposition Press, New York, 1955. 479 pp.

p. 231. Gauss used to say that he was entirely a mathematician, and he rejected the desire to be anything different at the cost of mathematics. It is true that the research in physical science offered him a type of recreation. He called mathematics the queen of the sciences, and the theory of numbers the queen of mathematics, saying that she often condescended to serve astronomy and other sciences, but that under all circumstances top rank belonged to her. Gauss regarded mathematics as the principal means of educating the human mind. He recognized the value of studying classical literature, and said that although he chose mathematics as a career he had not neglected the latter. Gauss recommended to his students the study of ancient mathematicians, in particular Euclid and Archimedes.



$A \cap B$, where

$$A = \left\{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$$

$$B = \left\{(x,y,z) : \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}.$$

A. R. Amir-Moéz

ON THE ELEMENTARY APPROACH TO DIOPHANTINE EQUATIONS

Oliver Gross

1. DISCUSSION

The primary purpose of this paper is to advocate the use of elementary methods in the solution of equations in integers insofar as such methods can be exploited. When they fail the writer has no objection to the use of more advanced techniques. By advanced, here, as opposed to elementary, we do not necessarily mean deep or complicated. What we do mean, rather, is that the techniques embody the assumption that other entities than the integers exist and that the integers themselves can somehow be embedded among them. We cite the use of algebraic number fields and contour integrations as specific examples.

It should of course be pointed out that the writer is not unique in his stand, insofar as others have advocated the exploitation of elementary methods as a complement to the use of advanced techniques. Nor do we assert here that the examples we present in the next section are in any way novel or complete in either results or methods.

Our espousal of the elementary approach can be expressed in a negative sort of way, namely, it is a sin of neglect to restrict oneself to the use, say, of algebraic number fields to the almost total exclusion of elementary number theory, especially when the use of the latter may be even superior to the former. A recent paper [1] by Skolem, Chowla and Lewis is a case in point. The writer has no argument against the elegant manner in which the authors of that paper disposed of Ramanujan's conjecture about the Diophantine equation $2^{n+2}-7 = x^2$ and doubts that their proof can be readily elementarized. However, in their Section 4, the authors state a "small generalization" of Ramanujan's problem:

"If A is an odd rational integer incongruent to 1 modulo 8, the equation $2^n + A = x^2$ has at most one rational integer solution for (n, x). If there is a solution, then $0 \leqq n \leqq 2$." (It is understood here, of course, that x and $-x$ are not to be distinguished.)

The proof they give begins as follows:

"Let $\mathbf{D}$ denote the ring of integers in $Q(A^{1/2})$, etc." Though the proof is fairly short, it involves the use of primes in $\mathbf{D}$, and is in fact longer than an elementary proof of the same assertion. At the expense, perhaps, of making the theorem appear too trivial, the proof would have reached more readers had it read as follows:

"Certainly if there is a solution, then $0 \leqq n \leqq 2$ (obviously n cannot be negative); for if $n \geqq 3$, then, since A is odd, x is odd and since an odd square is congruent to 1 modulo 8, we would have $A \equiv 1 \pmod 8$, contrary to hypothesis. The uniqueness question is now readily resolved. For if we had two solutions, (n_1, x_1), (n_2, x_2) with, say, $0 \leqq n_1 < n_2 \leqq 2$, we would obtain by subtracting the two equations

$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) = 2^{n_2} - 2^{n_1} = 1, 2, \text{ or } 3.$$

Thus, the remainder of the proof, in effect, involves factoring the integers 1, 2 and 3. We leave this last as an exercise to the reader."

Due to some quirk (or deficiency) of the human mind, perhaps, it is apparently easier at times first to prove a generalization in order to establish some result on a more elementary level, paradoxical as this may seem. Thus, the Kummerian method and its extensions have been applied with this philosophy in view, in attempts to prove Fermat's Last Theorem, admittedly with some success. Kummer's Theorem represents, of course, a generalization of a family of special cases of Fermat's Last Theorem. However, it is known that there are infinitely many non-Kummerian primes, and, to the writer's knowledge, it is conceivable that there are but finitely many regular ones. What is needed here, if one were safely to pursue the Kummerian approach, is a proof of the equivalence of Fermat's Last Theorem and its extension to cyclotomic number fields; but, again, to the writer's knowledge, such a proof does not exist. Indeed, one might conjecture that such a proof would perhaps be as difficult as a proof of the "theorem" itself. Moreover, one may well surmise that Fermat truly had a proof of his assertion, as he claimed; but if so, it appears doubtful, to say the least, that his method embodied an embedding of the integers in this peculiarly sophisticated manner.

At any rate, the point we wish to stress here is the inherent danger of "proofs by generalization": The proposed generalization of the assertion we wish to prove may well be false, or at best unprovable, even though the original assertion be capable of a proof (however elusive it might temporarily appear).

As a final clinching argument in favor of the use of elementary techniques, whether or not in conjunction with algebraic and/or analytic methods, we cite the important work of Julia Robinson [2] entitled "The Undecidability of Algebraic Rings and Fields." The final theorem of that paper reads as follows:

"*Theorem.* If F is an algebraic field of finite degree over the rationals, the natural numbers are arithmetically definable in F and hence F is undecidable."

The point we wish to bring out here is that neither the elementary approach nor, e.g., the algebraic approach can be used separately to solve arbitrary systems of Diophantine equations. One can at best use them in conjunction, to encompass a greater variety of problems, using more advanced techniques when the more elementary ones seem to fail (or vice versa).

Before embarking on the next section, we mention a pair of elementary "tricks" that are sometimes useful in treating systems of Diophantine equations. The first of these we employed earlier, namely, "An odd square is congruent to 1 modulo 8." One need not run through the residue classes modulo 8 to observe this, as it is immediately evident from the identity,

$$(2k+1)^2 = 4k(k+1)+1 .$$

The second "trick" is the well-known fact that an integer of the form y^2+1 cannot have a positive integer divisor of the form $4k+3$. Perhaps the easiest way to prove this is by using Fermat's Theorem in conjunction with the fundamental theorem of arithmetic. Suppose, then, that y^2+1 has a positive divisor k of the form $4k+3$. Then d has a prime divisor of that form; for since d is positive, it is an odd integer > 1 and therefore has a prime divisor. But each of its prime divisors is of the form $4k+1$ or $4k+3$, and they cannot all be of the former form, for if we multiply them together to obtain d we would have d of that form, a contradiction. Hence, we have

$$y^2 \equiv -1 \pmod{p = 4k+3} .$$

Clearly, y is prime to p, and hence by Fermat's Theorem,

$$y^{p-1} \equiv y^{4k+2} \equiv 1 \pmod p .$$

On the other hand, raising both members of the previous congruence to the $2k+1$ power, we obtain

$$y^{4k+2} \equiv -1 \pmod p ,$$

a contradiction.

A considerable variety of equations can be resolved by the use of congruences alone. This point has already been noted by previous writers and will not be enlarged upon. Suffice it to say that the two tricks mentioned earlier together with this and others are illustrated in the examples treated in the following final section. So without further ado, we embark upon it.

2. *EXAMPLES*

As the first of our handful of examples, we consider the possibility of solving in integers the equation, $x^6+3 = y^3+z^3$. It is not necessary to hunt about for a suitable algebraic number field to resolve this question, since the solvability of the equation implies the solvability of the congruence,

$$x^6+3 \equiv y^3+z^3 \pmod 7 .$$

Now, the cubic residues of 7 are 0, 1, and -1 and hence the sum of two cubes cannot be congruent to 3 or 4; but we see that the left member of the congruence is always congruent to 3 or 4 mod 7. Thus, the equation has no solution in integers.

Since $\sqrt{3}$ is irrational, we know that the only solution to the equation $a^2 = 3b^2$ is given by $a = b = 0$. But this is of course what we mean when we say that $\sqrt{3}$ (if it exists) is irrational, and in fact we do not need to resort to $Q(\sqrt{3})$, say, in order to treat the more general equation,

$$a^2 + b^2 = 3(c^2 + d^2) .$$

One readily checks that if the sum of the squares of two integers is divisible by 3 only if each integer is divisible by 3. Thus, $a = 3\alpha$ and $b = 3\beta$. Substituting in the equation and dividing through by 3, we obtain virtually the same equation,

$$c^2 + d^2 = 3(\alpha^2 + \beta^2) .$$

Continuing in this manner, we see that each of the numbers a, b, c, d must be divisible by an arbitrarily high power of 3, and hence they must all be zero. In effect, we have obtained by Fermat's famous method of descent that the only solution to the equation is the trivial one, $a = b = c = d = 0$. More generally, if A is a positive integer, then a necessary and sufficient condition that the equation $a^2 + b^2 = A(c^2 + d^2)$ have a non-trivial solution is that the exponent of the highest power of every prime of the form $4k+3$, $(k \geqq 0)$ dividing A be even. This last is only slightly more difficult to prove, using the same line of attack and well-known elementary results, and is therefore left as an exercise.

Our next example is a special case of the so-called "Mordell's Equation,"

$$x^3 = y^2 + 3 .$$

Clearly, x must be odd, for otherwise we would have

$$y^2 + 3 \equiv 0 \pmod 8 ,$$

which is impossible. Moreover, the equation can be written, by adding 1 to each side and factoring the left member,

$$(x+1)(x^2 - x + 1) = y^2 + 4 .$$

Now, since x is odd, we have $x \equiv \pm 1 \pmod 4$. But we cannot have $x \equiv 1 \pmod 4$, for this would imply the impossible congruence $y^2 \equiv 2 \pmod 4$. Thus $x \equiv -1 \pmod 4$ and the second factor in the left member of the above equation is therefore a positive integer of the form $4k+3$. By a previous argument it has a prime divisor p of the same form, and we obtain

$$y^2 + 4 \equiv 0 \quad (\text{mod } p = 4k+3) .$$

Now, since $(2, p) = 1$, 2 has an inverse (mod p). Multiplying the congruence through by the square of its inverse, we obtain a congruence of the form

$$u^2 + 1 \equiv 0 \quad (\text{mod } p = 4k+3) .$$

Applying our second "trick" then shows that the equation has no solution in integers.

Next on our list is the quintic equation,

$$x^5 + x - 1 = y^2 .$$

We assert that the only solutions in integer pairs (x, y) are given by $x = 1$,

$y = \pm 1$. We observe that the equation can be written in factored form,

$$(x^2 - x + 1)(x^3 + x^2 - 1) = y^2 .$$

Moreover, from the readily verifiable identity,

$$(x^2 - x + 1)(x^2 + 4x + 5) - (x^3 + x^2 - 1)(x + 2) = 7 ,$$

we conclude that

$$(x^2 - x + 1,\ x^3 + x^2 - 1) = 1 \text{ or } 7 .$$

There is no loss in generality in assuming $x > 0$, so that in the latter case, each factor must be 7 times a square:

$$x^2 - x + 1 = 7\alpha^2$$

$$x^3 + x^2 - 1 = 7\beta^2 .$$

But from the original equation we see that y must be odd, and hence so are α and β, But since "an odd square is congruent to 1 modulo 8," the two equations imply the simultaneous congruences,

$$\left.\begin{aligned} x^2 - x + 1 &\equiv -1 \\ x^3 + x^2 - 1 &\equiv -1 \end{aligned}\right\} \pmod 8 .$$

However, a straightforward elimination of x yields the impossible congruence

$$4 \equiv 0 \pmod 8 .$$

Thus, the two factors must be relatively prime and hence each is a square. In particular, we must have an integer u such that

$$x^2 - x + 1 = u^2 .$$

Upon multiplying this last equation through by 4 and doing other minor manipulations, we obtain a rather trivial equation,

$$(2u + 2x - 1)(2u - 2x + 1) = 3 ,$$

which yields $x = 1$ as the only positive integer candidate, and hence $y = \pm 1$. We note in passing that an almost entirely analogous approach yields the following generalization:

If $n \equiv 1 \pmod 4$ and $12n^2 - 6n + 1$ is a prime, then the only solution in positive integers to the Diophantine equation

$$x^{6n-1} + x - 1 = y^2$$

is given by $x = 1$, $y = 1$. (E. g., the hypothesis is satisfied in particular by $n = 1, 5, 9, 13, 21$ and 25.) We toss this off as another exercise.

The Diophantine equation,

$$x^3 + x^2 + x = y^2 + y ,$$

seems to have some bearing on the impossibility of the existence of a three-dimensional finite projective space consisting of the same number of points as some finite projective plane. However, there is, e. g., a four-dimensional finite projective space of order 2, having exactly 31 points, namely, the one with homogeneous point (or hyperplane) coordinates $(x_1, x_2, x_3, x_4, x_5)$ in GF [2]. The number of points in the finite projective plane of order 5 is, or course, 31. We mention this as an aside to justify the treatment of the equation, whose solution, nonetheless, turns out to be completely trivial.

If we rewrite the equation in the form

$$x^3 = (y-x)(y+x+1) ,$$

we observe that the two factors in the right member are necessarily relatively prime, since any prime divisor of the first factor must divide the left member, hence it must divide x and so it must divide y, leaving therefore a remainder of 1 in the other factor. Thus each factor is a cube:

$$y - x = \alpha^3$$
$$y + x + 1 = \beta^3 ,$$

whence

$$x = \alpha\beta .$$

Eliminating x and y in the obvious manner gives the equation

$$2\alpha\beta + 1 = \beta^3 - \alpha^3 = (\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2) .$$

So that on taking absolute values, we obtain

$$|2\alpha\beta+1| = |\beta-\alpha|(\beta^2+\alpha\beta+\alpha^2) .$$

Clearly, $\beta \neq \alpha$ so that $|\beta - \alpha| \geqq 1$; moreover, we see from the original equa- that $x \geqq 0$. Consequently, we obtain the inequality

$$2\alpha\beta + 1 \geqq \beta^2 + \alpha\beta + \alpha^2 ,$$

or

$$1 \geqq \beta^2 - \alpha\beta + \alpha^2 .$$

Since the right member of this inequality is a positive definite quadratic form, it cannot vanish unless $\alpha = \beta = 0$. We have already ruled this out. Consequently,

$$\beta^2 - \alpha\beta + \alpha^2 = 1 ,$$

and we obtain from our eliminant that

$$\beta = \alpha + 1 .$$

The solution of this last pair of equations yields

$$x = \alpha\beta = 0 .$$

Whence, $y = 0$ or -1.

As our final example, we consider an equation with a variable exponent. The treatment is a bit more tedious than that of any of its predecessors, though unencumbered with the use of cylotomic numbers. We assert that the complete solution in positive integers (x, y, n) of the equation

$$x^n = y^2 + 2^n + 1$$

is given by the table:

x	y	n
k^2+3	k	1
3	2	2
3	8	4

We can, of course, dispense with the first line of the table and assume $n > 1$. Our first observation is that x must be odd. For since $n \geqq 2$, we would otherwise have $y^2 + 1 \equiv 0 \pmod 4$, an impossibility. We observe next that n must be even. For if $n = 2k+1$, $k > 0$, we would obtain

$$(x-2)(x^{2k-1} + 4x^{2k-2} + \cdots) = y^2 + 1 .$$

But since x is odd, $x \equiv \pm 1 \pmod 4$, and hence one of the two factors of the left member will be of the form $4k+3$. Since both factors are necessarily positive, our second trick rules this out. Thus, $n = 2m$ for some positive integer m, and our equation becomes

$$x^{2m} = y^2 + 4^m + 1 .$$

Whence, we obtain

$$(x^m - y)(x^m + y) = 4^m + 1 .$$

Taking absolute values and noting that $|x^m - y| \geqq 1$, we obtain

$$x^m \leqq |x^m + y| \leqq 4^m + 1 .$$

Thus, $x \leqq 4$. But since x is odd and clearly $\neq 1$, it follows that $x = 3$. Reverting back to our original exponent, we obtain as our equation

$$3^n - 2^n = y^2 + 1 .$$

We note next that n cannot have an odd divisor > 1, for if $n = q \cdot d$ with $d > 1$ and odd, we would have a factorization

$$(3^d - 2^d)(\ldots) = y^2 + 1$$

and again, the first factor of the left member would be a positive integer of the form $4k+3$. Thus, n must be a power of 2; i.e., $n = 2^k$, and our equation reduces to

$$3^{2^k} - 2^{2^k} - 1 = y^2 .$$

But this last equation implies, in particular, the congruence

$$3^{2^k} - 2^{2^k} - 1 \equiv y^2 \pmod{29} .$$

We can compute the residues 3^{2^k}, 2^{2^k} by successive squaring and reducing modulo 29 to obtain the table:

k	3^{2^k}	2^{2^k}	$3^{2^k}-2^{2^k}-1$
0	3	2	
1	9	4	
2	23	16	
3	7	24	11
4	20	25	
5	23	16	
6	7	24	11

We observe from our table and the method of generation that a cycle of length 3 commences at $k = 2$. Therefore if $k > 2$ and $\equiv 0 \pmod 3$, we would have $y^2 \equiv 11 \pmod{29}$. But one verifies that 11 is not a quadratic residue of 29. So we cannot have a solution in this case. In order to put a bound on k, therefore, we need only rule out $k \equiv 1$ or $2 \pmod 3$ for k sufficiently large. To this end we use the prime modulus 449 in the same fashion to obtain the table

k	3^{2^k}	2^{2^k}	$3^{2^k}-2^{2^k}-1$
0	3	2	
1	9	4	
2	81	16	
3	275	256	
4	193	431	
5	431	324	
6	324	359	
7	359	18	340
8	18	324	142
9	324	359	

One readily verifies, using the powerful quadratic reciprocity law (to expedite matters) that 340 and 142 are both non-residues of 449. We leave this last as an exercise. Having reduced our possibilities to a known small finite set it is now a trivial computation to verify that our table represents the totality of solutions to our original equation.

Before closing, the writer would like to answer a possible objection to his treatment of the foregoing examples. A number theorist totally addicted to non-elementary techniques might be tempted to believe that the examples were constructed in a devious way so as to be completely amenable to elementary techniques. Indeed they were.

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HE STRUCK OUT

Johnny insisted it was his right,

Cancelling every pair in sight,

Working his IGNMEY,

LGEBR, and GOMTRY.

Marlow Sholander

A NOTE ON THE LIMIT OF $f(x)/f'(x)$

Paul Schaefer

The functions $1/\sqrt{x}$, $\cot x$ and $\log x$ display the following rather surprising behavior as $x \to 0^+$: $|f(x)| \to +\infty$ and $f(x)/f'(x) \to 0$. However, for

$$f(x) = (x^2 \sin \frac{1}{x})^{-1},$$

$|f(x)| \to +\infty$ while $f(x)/f'(x)$ has no limit. This paper gives a sufficient condition for the limit of $f(x)/f'(x)$ to be zero when $|f(x)|$ increases without bound on some finite interval.

Lemma: If $|f'(x)|$ is monotonically decreasing on the interval (a, b),

$$\text{if} \quad \lim_{x\to a^+} |f(x)| = +\infty, \quad \text{then} \quad \lim_{x\to a^+} |f'(x)| = +\infty .$$

Proof: Let $c \in (a, b)$. For $x \in (a, c)$, $f(c)-f(x) = (c-x)f'(\xi)$ where $x<\xi<c$. Then $|f(x)| \leqq (c-x)|f'(\xi)| + |f(c)|$. If $\lim_{x\to a^+} |f'(x)| \neq +\infty$, then for every δ, where $0 < \delta < (c-a)$, there exist $G>0$ and $t \in (a, a+\delta)$ such that $|f'(t)| \leqq G$. Then $|f(t)| \leqq (c-t)|f'(\xi)| + |f(c)| \leqq (c-t)|f'(t)| + |f(c)| \leqq (c-a)G + |f(c)|$, which is a contradiction.

Theorem 1: If $|f'(x)|$ is monotonically decreasing on the interval (a, b),

$$\text{if} \quad \lim_{x\to a^+} |f'(x)| = +\infty, \quad \text{then} \quad \lim_{x\to a^+} f(x)/f'(x) = 0 .$$

Proof: Let $\epsilon > 0$ be given. Choose $c \in (a, a+\frac{\epsilon}{4})$. Let $x \in (a, c)$. Then $f(c)-f(x) = (c-x)f'(\xi)$ where $x < \xi < c$, so that $|f(x)| \leqq (c-x)|f'(\xi)| + |f(c)|$. Since $a < x < c < a+\frac{\epsilon}{4}$, $0 < c-x < \frac{\epsilon}{4}$. Also, $|f'(\xi)| \leqq |f'(x)|$. Therefore $|f(x)| \leqq \frac{\epsilon}{4} |f'(x)| + |f(c)|$. Choose δ, $0<\delta<(c-a)$, so that when $a < x < a+\delta$, $|f'(x)| > 2|f(c)|/\epsilon$. Then for these x,

$$|f(x)/f'(x)| < \frac{\epsilon}{4} + |f(c)/f'(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon .$$

The next theorem follows directly from the lemma and theorem 1:

Theorem 2: If $|f'(x)|$ is monotonically decreasing on the interval (a, b),

$$\text{if} \quad \lim_{x\to a^+} |f(x)| = +\infty, \quad \text{then} \quad \lim_{x\to a^+} f(x)/f'(x) = 0 .$$

These results hold for left-hand limits at $x = b$, with appropriate changes in the proofs, when the condition "$|f'(x)|$ is monotonically decreasing" is replaced by "$|f'(x)|$ is monotonically increasing."

State University of New York
College of Education at Albany

STRADDLES ON SEMIGROUPS

S. P. Franklin and John W. Lindsay

In the study of modern (abstract) algebra, the theory of groups is of fundamental importance. The properties of groups have been well explored, and an introduction to the subject may be found in [1]. Somewhat weaker systems have been studied, and among these is the semigroup.

According to Jacobson [2], a *semigroup* is a system consisting of a set S and an associative binary composition in S. Thus, if $\cdot$ is the binary composition in S, for any two elements x and y in S, their product $x \cdot y$ (or simply xy) is also in S, and this composition obeys the associative law.

DEFINITION: In a semigroup S, the ordered pair (a, b) is a *straddle on* S if, and only if, a and b are in S and for every element x in S, $axb = x$. Whenever $axb = x$, we shall say a and b straddle x.

Example 1. Consider the set $S = \{1, -1, i, -i\}$, where $i^2 = -1$, and the usual rules of multiplication apply. Then i and $-i$ straddle every element of S, hence $(i, -i)$ is a straddle on S. However (i, i) is not a straddle; the straddle $(1, 1)$ is called the trivial straddle.

If a semigroup S has an element e such that $xe = ex = x$ for all x in S, then e is called an identity for S. Clearly a semigroup with an identity e has the straddle (e, e), the trivial straddle.

THEOREM 1. A semigroup with a non-trivial straddle has an identity. In fact, if (a, b) is a straddle on S, then $ab = e$ is the identity for S.

Proof: By definition, if (a, b) is a straddle on S, then $a = aab$. Hence, for any element x,

$$x = axb = (aab)xb = a(abx)b = abx \ .$$

Similarly, $b = abb$ and

$$axb = ax(abb) = a(xab)b = xab \ .$$

Hence, $(ab)x = x(ab) = x$, and $e = ab$ is the identity element.

COROLLARY 1. If (a, b) is a straddle on S, then (b, a) is a straddle on S.

Proof: $bxa = a(bxa)b = (ab)x(ab) = x$.

COROLLARY 2. If (a, b) is a straddle on S, then a and b commute with every element of S.

Proof: By Theorem 1 and Corollary 1, we have (b, a) is a straddle and $ab = e = ba$. Let x be any element of S. Then

$$bx = a(bx)b = (ab)xb = xb \ .$$

Similarly, $ax = b(ax)a = xa$.

If an element a commutes with every element of S we shall say that a is a *commutative* element of S. In a semigroup S with an identity e, we shall call an element a of S *right-regular* if there is an element a' of S such that $aa' = e$, and a' is called the *right inverse* of a. Similarly for *left-regular* and *left inverse*. (See [2]).

Lemma. In a semigroup S with identity, if a is a right-regular commutative element, with right inverse a', then (a, a') is a straddle on S

Proof: Let x be an element of S. Then $axa' = xaa' = xe = x$. From the foregoing we immediately have

THEOREM 2. A semigroup S has a non-trivial straddle if and only if S has an identity and a right-regular commutative element other than the identity.

Remark. The hypothesis of the lemma can be weakened to require only a right-regular element such that every element commutes with either a or a'.

THEOREM 3. If S is a semigroup with a straddle on S, then the set of straddles on S is a commutative group under the product operation $\circ$ defined by

$$(a, b) \circ (c, d) = (ac, bd) .$$

Proof: Because of the commutative properties of straddle elements, the product of two straddles is a straddle, and the associative and commutative laws follow immediately. (e, e) is the identity straddle, and the inverse of (a, b) is (b, a).

Example 2. Let $S = \{a, b, c\}$, with composition (multiplication) given by the following table:

$\cdot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

The associative law is easily verified, and thus S is a semigroup. However, S is not a group since c has no inverse. The trivial straddle is (a, a), and (b, b) is the only non-trivial straddle; hence $\overline{S} = \{(a, a), (b, b)\}$ is the straddle group.

Remark: If H is the group of inner-automorphisms (see [1]) of a group G, then the straddle group is a subgroup of H.

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University of California at Los Angeles

A SIMPLE ITERATION ALGORISM FOR GENERATING

$\cos nx$ AND $\sin nx$

Edgar Karst

With the increasing use of electronic computers, algorisms based on iteration become more and more important. Moreover, iteration methods are often much simpler than the usual calculating methods, and results can be obtained much easier and quicker.

Our first task shall be to extend the table of the first six $\cos nx$ to the next six. The first six $\cos nx$ are:

$$\cos 1x = 1\cos x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 3x = 4\cos^3 x - 3\cos x$$
$$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$$
$$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$$
$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

Now, if we use the usual calculating method, we have to get each further $\cos nx$ by means of the formula $\cos(x+y) = \cos x \cos y - \sin x \sin y$, which is still useful in occasional checking, but is cumbersome, because many values calculated previously will vanish and all $\sin^k x$ have to be expressed in terms of $\cos^k x$.

With logical thinking and a simple iteration algorism we get our evaluations much sooner. First we see that the first coefficients 1, 2, 4, 8, 16, and 32 follow the law 2^{n-1}, and we can write 64, 128, 256, 512, 1024, and 2048 in the spaces provided for them. Then we extend $\cos^k x$ and its further members in which the exponents vertically increase by one, but horizontally decrease by two. Furthermore, the constants -1 and $+1$ exist only in $\cos nx$ with n even and are alternating. Finally, the signs plus and minus are alternating horizontally, but are constant vertically, and only the coefficients remain to be found. These can be generated by the following iteration algorism.

Algorism 1: Second coefficients: $2\cdot 1+2^0 = 3$, $2\cdot 3+2^1 = 8$, $2\cdot 8+2^2 = 20$, $2\cdot 20+2^3 = 48$, $2\cdot 48+2^4 = 112$, $2\cdot 112+2^5 = 256$, $2\cdot 256+2^6 = 576$, $2\cdot 576+2^7 = 1280$, $2\cdot 1280+2^8 = 2816$, $2\cdot 2816+2^9 = 6144$. Third coefficients: $2\cdot 1+3 = 5$, $2\cdot 5+8 = 18$, $2\cdot 18+20 = 56$, $2\cdot 56+48 = 160$, $2\cdot 160+112 = 432$, $2\cdot 432+256 = 1120$, $2\cdot 1120+576 = 2816$, $2\cdot 2816+1280 = 6912$. Fourth coefficients: $2\cdot 1+5 = 7$, $2\cdot 7+18 = 32$, $2\cdot 32+56 = 120$, $2\cdot 120+160 = 400$, $2\cdot 400+432 = 1232$, $2\cdot 1232+1120 = 3584$. Fifth coefficients: $2\cdot 1+7 = 9$, $2\cdot 9+32 = 50$, $2\cdot 50+120 = 220$, $2\cdot 220+400 = 840$. Sixth coefficients: $2\cdot 1+9 = 11$, $2\cdot 11+50 = 72$.

Therefore, our extended $\cos nx$ table will look:

$$\cos 7x = 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$$

$$\cos 8x = 128\cos^8 x - 256\cos^6 x + 160\cos^4 x - 32\cos^2 x + 1$$

$$\cos 9x = 256\cos^9 x - 576\cos^7 x + 432\cos^5 x - 120\cos^3 x + 9\cos x$$

$$\cos 10x = 512\cos^{10} x - 1280\cos^8 x + 1120\cos^6 x - 400\cos^4 x + 50\cos^2 x - 1$$

$$\cos 11x = 1024\cos^{11} x - 2816\cos^9 x + 2816\cos^7 x - 1232\cos^5 x + 220\cos^3 x - 11\cos x$$

$$\cos 12x = 2048\cos^{12} x - 6144\cos^{10} x + 6912\cos^8 x - 3584\cos^6 x + 840\cos^4 x - 72\cos^2 x + 1$$

Coefficients (inclusive sign) of each $\cos nx$ add up to 1.

Now we will try to do the same with $\sin nx$. But here it is more difficult, as the first six $\sin nx$ show:

$$\sin 1x = 1\sin x$$

$$\sin 2x = 2\sin x\cos x$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\sin 4x = 8\cos^3 x\sin x - 4\cos x\sin x$$

$$\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$$

$$\sin 6x = 32\cos^5 x\sin x - 32\cos^3 x\sin x + 6\cos x\sin x$$

First we see that each $\sin nx$ with n odd contains the factor $\sin x$, and each $\sin nx$ with n even contains the factor $2\sin x\cos x = \sin 2x$. So, let us write them separately and only in terms of $\sin x$ and $\sin 2x$.

A: $\sin nx$ with n odd $= \sin(2m-1)x$:

$$\sin 1x = \sin x(1)$$

$$\sin 3x = \sin x(3 - 4\sin^2 x)$$

$$\sin 5x = \sin x(5 - 20\sin^2 x + 16\sin^4 x)$$

B: $\sin nx$ with n even $= \sin 2mx$:

$$\sin 2x = \sin 2x(1)$$

$$\sin 4x = \sin 2x(2 - 4\sin^2 x)$$

$$\sin 6x = \sin 2x(3 - 16\sin^2 x + 16\sin^4 x)$$

Now let us look only at the members within the parentheses. Obviously the constants in A follow the law n, and the constants in B follow the law $n/2$. Then we extend in A and B all $\sin^k x$, until the exponent of the last member of each single $\sin nx$ is higher by 2 than the exponent of the last member of $\sin(n-1)x$. All exponents increase by 2 horizontally, but are constant vertically. Finally, the signs minus and plus in A and B are alternating horizontally, but are constant vertically, and again only the coefficients remain to be found.

In observing the coefficients we see that those of the last members in

A and B follow the law 4^{m-1}. Therefore, only the coefficients between constants and last members in each single $\sin nx$ remain unknown. But here we get tremendous help from the table of $\cos nx$, which we just generated. Because the coefficients of $\cos nx$ with n odd, written from right to left, are the coefficients of A, inclusive constants and coefficients of last members, the extended table of A will look:

A: $\sin nx$ with n odd $= \sin(2m-1)x$ for $m = 4$, 5, and 6:

$$\sin 7x = \sin x(7 - 56\sin^2 x + 112\sin^4 x - 64\sin^6 x)$$

$$\sin 9x = \sin x(9 - 120\sin^2 x + 432\sin^4 x - 576\sin^6 x + 256\sin^8 x)$$

$$\sin 11x = \sin x(11 - 220\sin^2 x + 1232\sin^4 x - 2816\sin^6 x + 2816\sin^8 x - 1024\sin^{10} x)$$

Coefficients (inclusive sign) of each $\sin(2m-1)x$, with m odd, add up to 1, while coefficients (inclusive sign) of each $\sin(2m-1)x$, with m even, add up to -1.

Unfortunately we cannot apply this simple rule to the coefficients of B. But logical thinking leads us to the following algorism for generating the coefficients of B, exclusive constants and coefficients of last members.

Algorism 2: Take from $\sin nx$ with $n = 4$ the first coefficient (4 in $4\sin^2 x$). Subtract it (4) from the first coefficient of $\sin(n+1)x$ (which is 20 in $20\sin^2 x$). Set the result (16) as first coefficient into $\sin(n+2)x$. Are now all coefficients of $\sin(n+2)x$ known? Yes. Take the first coefficient of $\sin(n+2)x$ (which is 16 in $16\sin^2 x$). Subtract it (16) from the first coefficient of $\sin(n+3)x$ (which is 56 in $56\sin^2 x$). Set the result (40) as first coefficient into $\sin(n+4)x$. Are now all coefficients of $\sin(n+4)x$ known? No. Take the next coefficient of $\sin(n+2)x$ (which is 16 in $16\sin^4 x$). Subtract it (16) from the next coefficient of $\sin(n+3)x$ (which is 112 in $112\sin^4 x$). Set the result (96) as next coefficient into $\sin(n+4)x$. Are now all coefficients of $\sin(n+4)x$ known? Yes. Take the first coefficient of $\sin(n+4)x$, and so on.

With such a simple algorism, based only on the subtraction of one integer from another, we calculate the next coefficients easily, so that we can place the coefficients 80, 336, and 512 in the empty spaces of $\sin 10x$, and the coefficients 140, 896, 2304, and 2560 in the empty spaces of $\sin 12x$. Therefore, the extended table of B will appear as:

B: $\sin nx$ with n even $= \sin 2mx$ for $m = 4$, 5, and 6:

$$\sin 8x = \sin 2x(4 - 40\sin^2 x + 96\sin^4 x - 64\sin^6 x)$$

$$\sin 10x = \sin 2x(5 - 80\sin^2 x + 336\sin^4 x - 512\sin^6 x + 256\sin^8 x)$$

$$\sin 12x = \sin 2x(6 - 140\sin^2 x + 896\sin^4 x - 2304\sin^6 x + 2560\sin^8 x - 1024\sin^{10} x)$$

Coefficients (inclusive sign) of each $2mx$, with m odd, add up to m, while coefficients (inclusive sign) of each $2mx$, with m even, add up to $-m$.

Brigham Young University
Provo, Utah

PRIME PORTIONS OF 1961

C. W. Trigg

(1) $1961 = 37 \cdot 53 = 19(10) + 61$.

(2) $$1+9+6+1 = 17, \quad 1^2+9^2+6^2+1^2 = 7 \cdot 17,$$
$$1^3+9^3+6^3+1^3 = 947, \quad 1^4+9^4+6^4+1^4 = 7859,$$
$$1^5+9^5+6^5+1^5 = 17 \cdot 3931, \quad 1^6+9^6+6^6+1^6 = 277 \cdot 2087.$$

(3) $$1^1+9^2+6^3+1^4 = 13 \cdot 23,$$
$$1^4+9^3+6^2+1^1 = 13 \cdot 59.$$

(4) $$19^2+61^2 = 2 \cdot 13 \cdot 157, \quad 19^2+16^2 = 617,$$
$$91^2+61^2 = 2 \cdot 17 \cdot 353, \quad 91^2+16^2 = 8537,$$
$$11^2+69^2 = 2 \cdot 2441, \quad 11^2+96^2 = 9337.$$

(5) $$19^2-16^2 = 3 \cdot 5 \cdot 7, \quad 96^2-11^2 = 5 \cdot 17^2.$$

(6) $$\begin{aligned} 1961 &= 1+977+983 \\ &= 2+647+653+659 \\ &= 5+479+487+491+499 \\ &= 5+311+313+317+331+337+347 \\ &= 61+223+227+229+233+239+241+251+257 \\ &= 47+167+173+179+181+191+193+197+199+211+223 \\ &= 43+131+137+139+149+151+157+163+167+173+179+181+191 \\ &= 5+83+89+97+101+103+107+109+113+127+131+137+139+149 \\ &\quad +151+157+163 \\ &= 13+53+59+61+67+71+73+79+83+89+97+101+103+107+109 \\ &\quad +113+127+131+137+139+149 \\ &= 1+13+17+19+23+29+31+37+41+43+47+53+59+61+67+71 \\ &\quad +73+79+83+89+97+101+103+107+109+113+127+131+137. \end{aligned}$$

In each of these series, all the primes but the first one are consecutive.

Los Angeles City College

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York*.

SUCCESSIVE DIFFERENTIABILITY

L. H. Lange

The ideas discussed here are not really new. It has merely been my experience that many calculus students find these ideas to be a great surprise and are consequently very attentive when they are discussed. Some have allowed themselves to believe that the traditional results of calculus are all intuitively quite "natural" and that even minor attempts at sophistication – for example, following or constructing a careful proof– are merely evidence of pedantry on the part of the teacher or weakness on the part of the student's intuition. Calculus made *too* easy, by too many significant omissions or by a lack of care in distinguishing proof from non-proof, can be a disservice. It is possible to stimulate respect for, and more than a little interest in, the ideal of rigorous procedure while still doing justice to the traditional formulae of calculus. Perhaps, then, this note about differentiability is not without merit.

A class in elementary differential equations may be discussing the Wronskian of n functions $f_1, f_2, \ldots, f_n$ defined on an interval $a \leqq x \leqq b$, and may be asked to "assume that each of these functions is differentiable at least $(n-1)$ times on this interval." Unless this hypothesis is in some way lightly passed over, it is not unusual for a question like this to arise: Can it be that a particular function, ϕ, could behave nicely when I take its first *six* derivatives and then misbehave on the seventh?![1]... If so, *how badly* could it then act?

Well now, let's be clear about what is meant. If we consider the function f, defined for all real x by $f(x) = x$, we have

$$f'(x) = 1\ ,\quad f''(x) = 0\ ;\quad f^{(n)}(x) = 0 \quad \text{for all } n \geqq 2\ .$$

This function is *not* an example of what we mean; it possesses a perfectly good derivative of every order.

Let's consider another function, g, given by $g(x) = |x|$ for all x in the

[1] In essence, these questions are at least 80 years old; Weierstrass dealt with them, of course.

interval $-1 \leqq x \leqq +1$. If we look at its graph we see immediately that for $x \neq 0$, *n. b.*, $g'(x) = |x|/x$ and that the number $g'(0)$ fails to exist; for

$$\lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^-} (-1) = -1 \neq +1 = \lim_{h\to 0^+} \frac{g(0+h)-g(0)}{h} .$$

Note: I didn't want to say here that "we can *see* the curve has no tangent at $x = 0$," for reasons which will be apparent later.

Now *several* questions can arise. *One*: returning to our function ϕ, above, could it be that the sixth derivative of ϕ, $\phi^{(6)}$, would be like our function g and that the reason $\phi^{(7)}$ doesn't exist is that for one particular value of x, say $x = 0$, the number $\phi^{(7)}(0)$ fails to exist? *Two*: if the answer to the first question is yes, could it be, and I hardly believe it possible (!), could it be that a function ϕ, defined on an interval $a \leqq x \leqq b$, would be such that $\phi^{(6)}(x)$ exists for all x in this interval, while $\phi^{(7)}(x)$ does not exist for *any* x in this interval?

The answer to both questions is, of course, yes. It is the affirmative answer to the *second* question which invariably seems to be a big surprise. I suppose the reason for this surprise is connected with the fact that unless a continuous curve we draw on a blackboard possesses a glaring cusp or two it seems intuitively obvious that the curve possesses a tangent at each of its points. *See Figure* II, *below, for example.*

To borrow a phrase, the teacher can "seize this moment of excited curiosity" to launch a discussion of some very important theorems in analysis. Here is a discussion which could follow, perhaps somewhat tersely presented in parts.

Returning to our function given by $g(x) = |x|$, above, it is easy to find a function G such that $G'(x) = g(x)$ for all x in question. We can, for example, recall the part of the *fundamental theorem of the calculus* — a part often slighted — which says that *any* continuous function, g, on a closed interval $[a, b]$ possesses an anti-derivative G, and that such a G can be calculated by the formula

$$G(x) = \int_a^x g(t)dt .$$

For the particular g we are considering, we may then take the G given by

$$G(x) = \begin{cases} \dfrac{1}{2} - \dfrac{x^2}{2}, & x < 0 \\ \\ \dfrac{1}{2} + \dfrac{x^2}{2}, & x \geqq 0 , \end{cases}$$

a result we can see geometrically, if we wish, by considering Figure I, where we must either subtract or add the area $\frac{x^2}{2}$ to the area, $\frac{1}{2}$, of triangle OAB, depending on the sign of x.

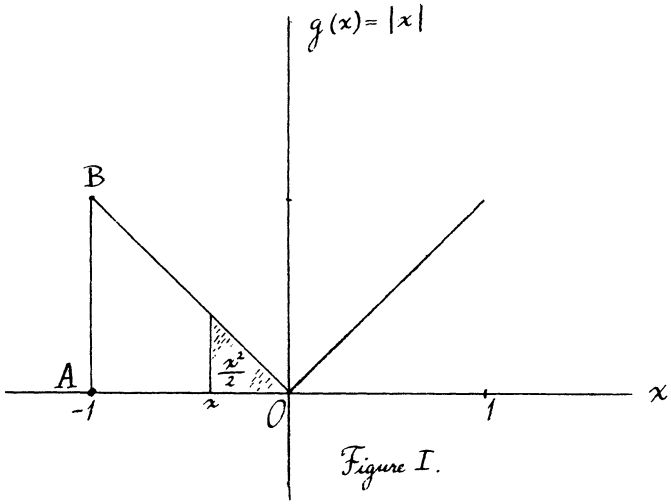


Figure I.

If we wish to dispose of the first question above with even more detail, we could easily use definitions and results like these: let x satisfy $-1 \leqq x \leqq +1$ and let

$$F_1(x) = g(x) = |x| \,,$$

$$F_2(x) = G(x) = \frac{1}{2} + (\operatorname{sgn} x) \cdot \frac{x^2}{2} \,;$$

where $\operatorname{sgn} x = -1$ if $x < 0$, and $\operatorname{sgn} x = +1$ if $x \geqq 0$;

$$F_3(x) = \frac{1}{2} x + (\operatorname{sgn} x) \cdot \frac{x^3}{3!} \,,$$

$$\cdots$$

$$F_n(x) = \frac{1}{2} \cdot \frac{x^{n-2}}{(n-2)!} + (\operatorname{sgn} x) \cdot \frac{x^n}{n!} \,.$$

$$\cdots$$

Thus we get a sequence of antiderivatives of F_1; *i. e.*, $F_n' = F_{n-1}$. One could then discuss the graphs of the functions F_n and even the uniformity of

$$\lim_{n \to \infty} F_n(x) = 0 \quad \text{in } [-1, +1] \,.$$

Now, to the second question above. The answer was given by Weierstrass in the latter part of the 19th Century and there are various ways of proceeding. Here I like to build on an example given by Professor J. M. H. Olmsted.[2]

---

[2] See (C) below. In addition to Olmsted's references to van der Waerden and to Titchmarsh, reference could be made to the related material in (A) and (B) of the bibliography below.

Let us construct a function f on [0, 1/2] which is *continuous*, (and therefore *has* an *antiderivative*) and which fails to have a *derivative* at any x in [0, 1/2]. To do this we define a certain sequence of functions, f_n, on [0, 1/2]; $n = 0, 1, 2, \cdots$. The functions f_n are "saw-tooth" functions on [0, 1/2], easily visualized, and given by the following rule:

$$f_0(x) = x \ ;$$

and for $n \geqq 1$,

$$f_n(x) = 0 \quad \text{for } x = \frac{k}{4^n}\text{, } k \text{ a positive integer or zero, and}$$

$$f_n\left(\frac{k}{4^n} + \frac{1}{2}\cdot\frac{1}{4^n}\right) = f_n\left(\frac{2k+1}{2\cdot 4^n}\right) = \frac{1}{2}\cdot\frac{1}{4^n}\,,$$

the values of $f_n(x)$ for intermediate values of x being determined by straight line segments of slope ± 1, joining the points so determined. Then, for each x in [0, 1/2], we define f by letting

$$f(x) = \sum_{n=0}^{\infty} f_n(x)\ .$$

Our function f is continuous because we may apply the following important theorem: the function defined by a *uniformly* convergent series of continuous functions is itself continuous. Our series of continuous functions is uniformly convergent since for any x in [0, 1/2], and any n, we have

$$0 \leqq f_n(x) \leqq \frac{1}{2}\left(\frac{1}{4^n}\right)$$

and

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{4^n}\right) < \infty\ .$$

Again, it is Weierstrass who has helped us, through his "M-test."

The construction of the proof of the fact that $f'(x)$ fails to exist for *any* x in [0, 1/2] is succinctly hinted at by Professor Olmsted. A carefully drawn figure helps the student greatly in understanding Olmsted's exposition of this fact.

An attempt to picture the function f so defined is shown in Figure II.

Finally, to answer completely our second question above, we observe that we can again make repeated use of the fundamental theorem of the calculus to start with our f and assert the existence of a sequence of antiderivatives: $\cdots, F_n, F_{n-1}, \cdots, F_1$ such that

$$F_1' = f \quad \text{and} \quad F_n' = F_{n-1} \quad \text{for } n \geqq 2 .$$

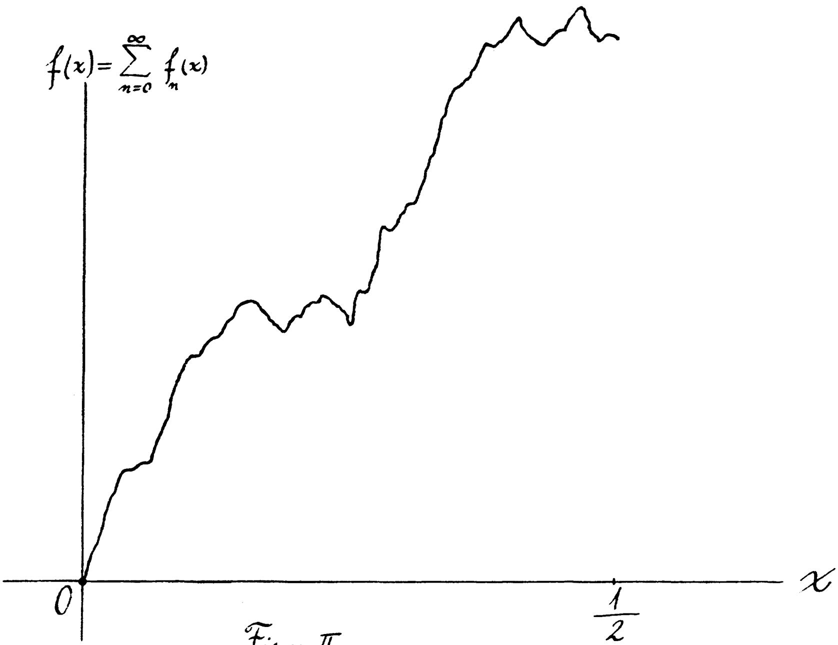


Figure II.

REFERENCES

(A) F. Hausdorff, *Set Theory*, Chelsea, 1957, pp. 230-235.

(B) E. Hille, *Analytic Function Theory*, Ginn, 1959, p. 35.

(C) J. M. H. Olmsted, *Intermediate Analysis*, Appleton-Century, 1956, pp. 286-287, Exercise 41.

San Jose State College
San Jose, California.

VECTOR ANALYSIS

When α did in β find
A magnitude of kindred bent
And they assumed the ties that bind,
They little dreamed what was designed
For bundles heaven sent.
Their union blessed, they first begot
A daughter by the name of Dot.
She was of lower order, not
Directed like her parents.
The second child to grace their house
A surly product, christened Kraus,
Was non-associative; a louse
Who needed much forbearance.
The third one, tenser and afflict'
With cravings of the dye addict,
In matrix forms we now restrict
To lessen his aberrance.
If this succinct analysis
Should bear a moral, it is this :
The claim that bliss must father bliss
Has negative transparence.

Marlow Sholander

TEACHING INVERSE TRIGONOMETRIC FUNCTIONS

Raymond S. Potter

Most books on trigonometry make the first step in teaching inverse trigonometric functions by interchanging function axes. The horizontal axis changes from being the angle designation to becoming the function designation. The vertical axis changes from being the value of the function to becoming the angle designation. In terms of mathematical symbols this means changing from $y = \sin x$ and $x = \arcsin y$ to $y = \arcsin x$ and $x = \sin y$.

These changes require the student to re-orient his thinking by going through the mental gymnastics of inverting all curves and rotating them counterclockwise through an angle of ninety degrees. The changes are perhaps good because they require the student to think and because the range of variation of a particular function is stressed as this range is now indicated more clearly as a portion or all of the horizontal axis. However, the necessity for understanding inverse trigonometric functions comes at a time in the student's life when he is saddled with a full load of other course work and anything which would clarify the situation would then be sincerely appreciated.

It has been proven to my own satisfaction to be more instructive to present the subject of this note without making this interchange of axes. All of the ideas of inverse trigonometric functions certainly can be explained by using the graphs of the trigonometric functions which follow from the definitions of the functions. For example, obtaining solutions of the equation $\sin x + \cos x = 1$ (one of the more advanced problems of inverse trigonometric functions) can easily be found as shown in Figure 1. This

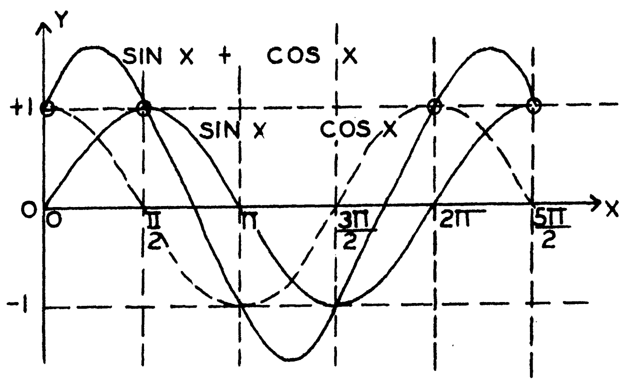


FIG. 1

is accomplished by sketching $\sin x$ and $\cos x$ versus x, then sketching the sum curve, and then obtaining the solutions of the equation as those angles

x which correspond to the points circled in Figure 1 for which the sum curve goes through $y = +1$. This easy way of obtaining the solutions to this equation may be missed entirely if the student is confused by the usual change of axes and the ensuing algebraic procedures which will locate the roots of the given equation and which are presented in all trigonometry textbooks.

Prince Georges Community College
Suitland, Maryland

ERRATA

In Volume 34, Number 4, March-April, 1961, on page 208 in the article by Verner E. Hoggatt, R_2 should read, $V_y = 2\pi I_2 = 2\pi\bar{x}A$. Also on page 209 in the same article R_6 should read, $M_y = I_4 = \bar{\bar{y}}V_x = \bar{\bar{y}}(2\pi\bar{y}A)$. On the same page, in 4.(e), the expression for $W(ork)$ should read: $W(ork) = WI_4 = \bar{\bar{x}}(WV_x)$.

A NOTE ON EQUATIONS AND INEQUALITIES

William A. Small

In most textbooks on elementary algebra, the steps followed to solve simple linear equations and inequalities are given separately, and it is usually stated that there is a difference in procedure for such solutions when negative multipliers or divisors are employed in solving inequalities. It is considered that this method can be improved upon, and this note serves to point out in what way this may be accomplished.

It will be shown herein that the *same* rules of procedure for solving linear inequalities apply as for solving linear equations, although in some cases the *results* may be different. That is, whether one is solving a linear equation or a linear inequality, he may use exactly the same rules of procedure in both cases, but he may sometimes obtain different results.

The old-style method of stating procedure for solving linear equations and linear inequalities will be outlined first, and then the improved method.

OLD STYLE

RULES FOR SOLVING LINEAR EQUATIONS

1. Equals may be added to or subtracted from equals, and the results remain equal.

2. Both sides of an equation may be multiplied or divided (by non-zero divisor) by the same number, and the results remain equal.

RULES FOR SOLVING LINEAR INEQUALITIES

1. Equals may be added to or subtracted from unequals, and the results remain unequal in the same order.
2. Both sides of an inequality may be multiplied or divided by the same non-zero number, and the results remain unequal: (a) in the same order if the multiplier (divisor) is positive; (b) in the reverse order if the multiplier (divisor) is negative.

Clearly, in the old-style rules, addition and subtraction are performed in the same way, whether one is working on an equation or an inequality. It remains to be seen how the multiplication and division rules may be restated to be the same in both cases.

This modification is as follows: When solving *either* an equation or an inequality, and when multiplying or dividing by a negative number, one *reverses* the sense of the relating sign, whether it is an equal sign, or an inequality sign. In the case of the equality, of course, nothing changes when the sign is reversed in sense. However, in the case of the inequality, the reversal of the sense of the inequality sign is noticeable. With this brief explanation, we may now state the new-style rules for the improved method.

NEW STYLE

RULES FOR SOLVING LINEAR EQUATIONS OR INEQUALITIES

COMMON RULE 1. Equals may be added to or subtracted from equals or unequals, and the results remain correspondingly equal or unequal in the same order.

COMMON RULE 2. Both sides of an equation or an inequality may be multiplied or divided by the same non-zero number, and the results remain correspondingly equal or unequal (a) in the same order if the multiplier (divisor) is positive; (b) in the reverse order if the multiplier (divisor) is negative.

COMMON RULE 3. Multiplication by zero reduces both an equation and an inequality to an identity: namely, $0 = 0$. This annuls the original equation or inequality.

Tennessee Polytechnic Institute
Cookeville, Tennessee

GEOMETRIC METAPHORS

"...You had to hand it to Jack Ribaldry; cornball though he was, he cut a wide swath. He may have been Harry Hypotenuse, the sum of the squares, but his spending arm never flagged. ..."

– S. J. Perelman, in The New Yorker, page 56, December 3, 1960.

Edited by

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

A NEW LOOK AT $e^{i\theta} = \cos\theta + i\sin\theta$

Henry Albaugh

Fig. 1 depicts geometrically the relationships we wish to use.

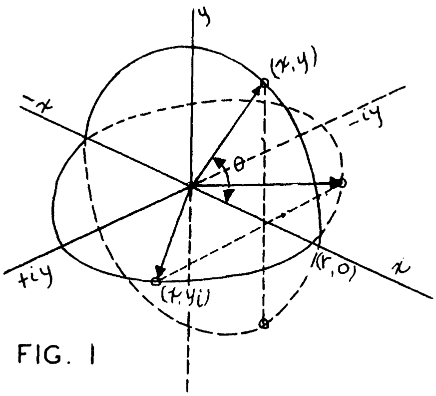


FIG. 1

From elementary analysis we may write

(1) $$x^2 + y^2 = r^2 \ .$$

Factoring (1),

(2) $$(x+iy)(x-iy) = r^2$$

(3) $$i = \sqrt{-1}$$

(4) $$r = |x \pm iy|$$

(5) $$\theta = \tan^{-1} y/x \ .$$

As the point (x, y) in the real plane moves in a counter-clockwise direction, the point (x, y_i) also moves in a corresponding direction in the complex plane.

We obtain by differentiation of (5),

(6) $$d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2} \ .$$

If we factor the denominator of (6) we get

$$d\theta = \frac{x\,dy - y\,dx}{(x+iy)(x-iy)} . \tag{7}$$

If we multiply both sides of (7) by $2i$, we obtain

$$2i\,d\theta = \frac{2xi\,dy - 2iy\,dx}{(x+iy)(x-iy)} . \tag{8}$$

The right side of (8) can be written

$$2i\,d\theta = \frac{2xi\,dy}{(x+iy)(x-iy)} - \frac{2iy\,dx}{(x+iy)(x-iy)} . \tag{9}$$

In accordance with the method of partial fractions, (9) may be expressed in the form

$$2i\,d\theta = \frac{Ai\,dy}{x+iy} + \frac{Bi\,dy}{x-iy} + \frac{C\,dx}{x+iy} + \frac{D\,dx}{x-iy} . \tag{10}$$

Hence

$$Ax - A(iy) + Bx + B(iy) = 2x .$$

Equating coefficients,

$$A+B = 2 , \quad -A+B = 0$$
$$B = 1 , \quad A = 1 .$$

Also,

$$Cx - C(iy) + Dx + D(iy) = -2(iy) .$$

Equating coefficients,

$$C+D = 0 , \quad -C+D = -2$$
$$C = 1 , \quad D = -1 .$$

So we find $A = 1$, $B = 1$, $C = 1$, $D = -1$.

Thus (10) becomes

$$2i\,d\theta = \frac{i\,dy}{x+iy} + \frac{i\,dy}{x-iy} + \frac{dx}{x+iy} - \frac{dx}{x-iy} . \tag{11}$$

Rearranging and combining (11) we have

$$2i\,d\theta = \frac{dx + i\,dy}{x+iy} - \frac{dx - i\,dy}{x-iy}$$

or

$$2i\,d\theta = \frac{d(x+iy)}{x+iy} - \frac{d(x-iy)}{x-iy} . \tag{12}$$

Hence by integration of (12)

$$\int 2i\,d\theta = \int \frac{d(x+iy)}{x+iy} - \int \frac{d(x-iy)}{x-iy}$$

or

$$2i\theta = \ln(x+iy) - \ln(x-iy) + \ln C_1. \tag{13}$$

If $\theta = 0$, $y = 0$, $\ln C_1 = 0$. Thus

$$2i\theta = \ln\left(\frac{x+iy}{x-iy}\right).$$

If we put (13) in the exponential form, we obtain

$$\frac{x+iy}{x-iy} = e^{2i\theta}. \tag{14}$$

If we multiply numerator and denominator of (14) by $(x+iy)$ we have

$$\frac{(x+iy)^2}{(x-iy)(x+iy)} = e^{2i\theta}$$

or, since $(x+iy)(x-iy) = r^2$,

$$\frac{(x+iy)^2}{r^2} = e^{2i\theta}. \tag{15}$$

Next we take the square root of (15) and we obtain

$$\frac{x+iy}{r} = e^{i\theta}$$

or

$$x+iy = re^{i\theta}. \tag{16}$$

(16) can also be written

$$\frac{x}{r} + i\frac{y}{r} = e^{i\theta}$$

or

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

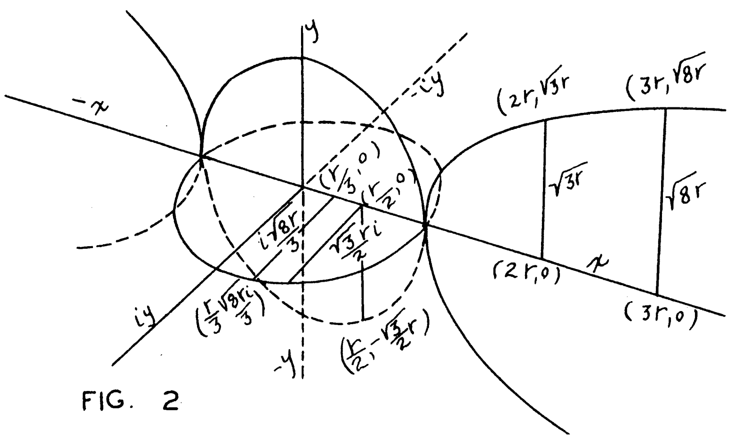


FIG. 2

If we take a new look at the equilateral hyperbola $x^2 - y^2 = r^2$ we see that we must use the imaginary plane for $x < r$. We write

(17) $$y_i = \pm\sqrt{r^2 - x^2}\, i \ .$$

Assign values to x. The following table helps visualize the corresponding geometric loci.

$$\boxed{x_i = \frac{r^2}{x}, \quad y_i = \frac{ry}{x} i}$$

$x \geq r$	x	$\pm r$	$\pm 2r$	$\pm 3r$	$\pm 4r$	x
	y	0	$\pm\sqrt{3}\, r$	$\pm\sqrt{8}\, r$	$\pm\sqrt{15}\, r$	y
$x \leq r$	x_i	$\pm r$	$\pm\frac{r}{2}$	$\pm\frac{r}{3}$	$\pm\frac{r}{4}$	$\frac{r^2}{x}$
	y_i	0	$\pm\frac{\sqrt{3}}{2} ri$	$\pm\frac{\sqrt{8}}{3} ri$	$\pm\frac{\sqrt{15}}{4} ri$	$\frac{ry}{x} i$

we see that

$$x_i^2 - y_i^2 = \frac{r^4}{x^2} + \frac{r^2 y^2}{x^2} = \frac{r^2(r^2 + y^2)}{x^2} = r^2$$

or

$$x_i^2 + (iy_i)^2 = r^2$$

then

(18) $$x_i^2 + \left(-\frac{yr}{x}\right)^2 = r^2 \ .$$

(18) demonstrates a correspondence between the points on the hyperbola and the points on a circle.

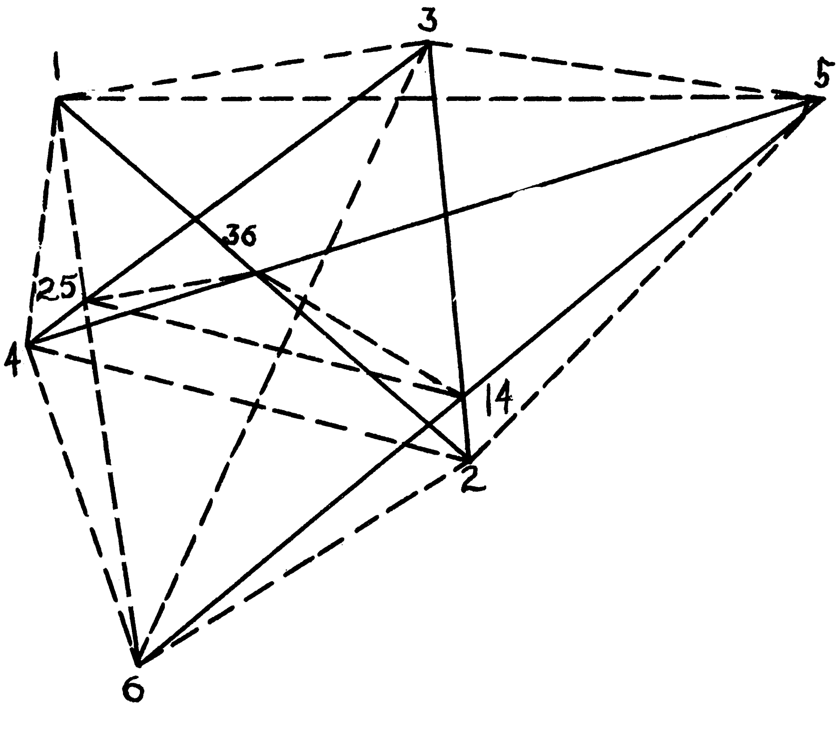
Hillsdale College
Hillsdale, Michigan

THE EXTENSION OF PASCAL'S THEOREM

C. E. Maley

Dedicated to the memory of Victor Thébault.

Six vertices $P_i(x_i, y_i)$ determine the simple hexagon or 6-point $P_1P_2P_3P_4P_5P_6$. Respective pairs of nonadjacent sides P_2P_3, P_5P_6; P_3P_4, P_6P_1; P_4P_5, P_1P_2 intersect in the Pascal points P_{14}, P_{25} and P_{36} determining the "Pascal triangle" $P_{14}P_{25}P_{36}$. The pairs of nonadjacent sides are also the diagonals of the "Pascal quadrilaterals" $P_1P_4P_2P_5$, $P_2P_5P_3P_6$ and $P_3P_6P_4P_1$, forming the complete 6-point.



Algebraically the six vertices establish the Vandermonde determinant of the quadratic simplex, of value in discussing conics and the error and rate of convergence of second order iterative methods [1]:

$$\phi(x^2,\ xy,\ y^2,\ x,\ y,\ 1)_1 \equiv \begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \end{vmatrix}.$$

It may be recalled that the vector area of triangle $P_{14}P_{25}P_{36}$ is

(1) $$\tfrac{1}{2}\begin{vmatrix} x_{14} & y_{14} & 1 \\ x_{25} & y_{25} & 1 \\ x_{36} & y_{36} & 1 \end{vmatrix}.$$

Maley [2] and Osborn [3] have extended this method to any polygon. For, since

$$\begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

is twice the vector area of the triangle OP_1P_4 then, by a process similar to contour integration,

(2) $$\begin{vmatrix} x_1 & y_1 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_5 & y_5 \end{vmatrix} + \begin{vmatrix} x_5 & y_5 \\ x_1 & y_1 \end{vmatrix}$$

is twice the vector area of the Pascal quadrilateral $P_1P_4P_2P_5$. It should be noted that the vector area of a quadrilateral may vanish although the scalar area does not.

In [2] it was indicated that method (2) immediately generalizes to volumes of polyhedrons (double integration). Here each determinant will contain the vertices of one triangular face written downward in the counterclockwise direction as viewed from outside the solid. (A nontriangular face may be decomposed into two or more triangular "faces".)

The intersection, if unique, of line P_1P_2,

(3) $$\phi(1,\ y)_1x + \phi(x,\ 1)_1y = \phi(x,\ y)_1\ ,$$

and line P_4P_5,

$$\phi(1, y)_4 x + \phi(x, 1)_4 y = \phi(x, y)_4 ,$$

where, for example,

$$\phi(x, y)_4 \equiv \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix},$$

will be the point P_{36}:

$$x_{36} = \frac{\begin{vmatrix} \phi(x, y)_1 & \phi(x, 1)_1 \\ \phi(x, y)_4 & \phi(x, 1)_4 \end{vmatrix}}{\begin{vmatrix} \phi(x, 1)_1 & \phi(y, 1)_1 \\ \phi(x, 1)_4 & \phi(y, 1)_4 \end{vmatrix}}, \quad y_{36} = \frac{\begin{vmatrix} \phi(x, y)_1 & \phi(y, 1)_1 \\ \phi(x, y)_4 & \phi(y, 1)_4 \end{vmatrix}}{\begin{vmatrix} \phi(x, 1)_1 & \phi(y, 1)_1 \\ \phi(x, 1)_4 & \phi(y, 1)_4 \end{vmatrix}}$$

The points P_{14} and P_{25} are similarly obtained.

By (1) and after reductions, the vector area, A, of the Pascal triangle is

$$2\begin{vmatrix} \phi(x, 1)_1 & \phi(y, 1)_1 \\ \phi(x, 1)_4 & \phi(y, 1)_4 \end{vmatrix}\begin{vmatrix} \phi(x, 1)_2 & \phi(y, 1)_2 \\ \phi(x, 1)_5 & \phi(y, 1)_5 \end{vmatrix}\begin{vmatrix} \phi(x, 1)_3 & \phi(y, 1)_3 \\ \phi(x, 1)_6 & \phi(y, 1)_6 \end{vmatrix} A =$$

$$(4) \qquad \begin{vmatrix} \begin{vmatrix} \phi(x, y)_1 & \phi(x, 1)_1 \\ \phi(x, y)_4 & \phi(x, 1)_4 \end{vmatrix} & \begin{vmatrix} \phi(x, y)_1 & \phi(y, 1)_1 \\ \phi(x, y)_4 & \phi(y, 1)_4 \end{vmatrix} & \begin{vmatrix} \phi(x, 1)_1 & \phi(y, 1)_1 \\ \phi(x, 1)_4 & \phi(y, 1)_4 \end{vmatrix} \\ \begin{vmatrix} \phi(x, y)_2 & \phi(x, 1)_2 \\ \phi(x, y)_5 & \phi(x, 1)_5 \end{vmatrix} & \begin{vmatrix} \phi(x, y)_2 & \phi(y, 1)_2 \\ \phi(x, y)_5 & \phi(y, 1)_5 \end{vmatrix} & \begin{vmatrix} \phi(x, 1)_2 & \phi(y, 1)_2 \\ \phi(x, 1)_5 & \phi(y, 1)_5 \end{vmatrix} \\ \begin{vmatrix} \phi(x, y)_3 & \phi(x, 1)_3 \\ \phi(x, y)_6 & \phi(x, 1)_6 \end{vmatrix} & \begin{vmatrix} \phi(x, y)_3 & \phi(y, 1)_3 \\ \phi(x, y)_6 & \phi(y, 1)_6 \end{vmatrix} & \begin{vmatrix} \phi(x, 1)_3 & \phi(y, 1)_3 \\ \phi(x, 1)_6 & \phi(y, 1)_6 \end{vmatrix} \end{vmatrix}.$$

The determinants on the lefthand side of (4) have a simple geometric meaning. For,

$$\begin{vmatrix} \phi(x, 1)_1 & \phi(y, 1)_1 \\ \phi(x, 1)_4 & \phi(y, 1)_4 \end{vmatrix} \equiv \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_4 - x_5 & y_4 - y_5 \end{vmatrix}$$

is precisely (2), twice the vector area of the quadrilateral $P_1P_4P_2P_5$.

The writer must confess that the righthand determinant was interpreted much less elegantly. The 512 terms of the main diagonal expansion were written down and the other 5 permuted diagonals were got by typing out, using six "caps" on the proper typewriter keys. (Perhaps the reader may devise a more elegant method. The writer believes there exists a calculus of determinants of which reference [2] and relation (4) are particular examples.) The resultant 3072 terms reduce to 720, so that

(5) $\phi(x^2, xy, y^2, x, y, 1)_1 = 16(P_1P_4P_2P_5)(P_2P_5P_3P_6)(P_3P_6P_4P_1)(P_{14}P_{25}P_{36})$.

Here $(P_1P_4P_2P_5)$, for example, indicates vector area.

Theorem. The Vandermonde determinant, $\phi(x^2, xy, y^2, x, y, 1)_1$, associated with any complete 6-point is equal to 16 times the product of the vector areas of the Pascal triangle and the 3 Pascal quadrilaterals.

Corollary. Pascal's Theorem of 1639.

Has the rich lode struck by Pascal 322 years ago been mined out? On the contrary, it has been strangely neglected.

Consider the well known theorem that the power of a point with respect to a circle, $\phi(x^2+y^2, x, y, 1)_1$, is equal to twice the product of the perpendicular distances from (x_1, y_1) to the circle determined by the other three points and the vector area of the inscribed triangle. Presented in this unorthodox form it suggests that the theorem of this article may be so expressed.

Again, the theorem may be dualized, by considering the Plücker incidence condition,

(3) $$ux + vy = 1 ,$$

that the point (x, y) lie on the line (u, v). Pascal has left many challenges, including the generalization of his theorem, and the present extension, to space of n dimensions.

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3. Osborn, Roger. "Some Geometric Considerations Related to the Mean Value Theorem." Mathematics Magazine, May-June, 1960.

The Carborundum Company
Niagara Falls, N. Y.

SOME OLD SLANTS AND A NEW TWIST TO THE CONE

Daniel B. Lloyd

When one considers the prodigious prolificity of theorems established by Apollonius and others upon the cone and the conic sections, it may be surprising to observe that certain aspects still remain unexplored. Although Apollonius (3rd century B.C.), a master of the synthetic method in geometry, established 387 propositions on the cone during his forty-year life time, his results have been extended by many others even down to modern times. In the latter part of this paper the author wishes to contribute one new development on the cone, but first proposes to invite the reader's attention to some earlier aspects of the cone's illustrious career.

The discovery of the conic sections is attributed to Menaechmus in the 4th century B.C. He employed conics to solve the famous Delian problem – the Duplication of the Cube. Apollonius investigated the non-focal properties of the conic, its conjugate diameters, asymptotes, and the harmonic property of the pole and polar for the case in which the pole lies outside the curve. Not until the 17th century were any further major advances made on conics. The latter theorem of Apollonius was then generalized by the genius of Desargues.

Among the lost works attributed to Euclid were four Books on *Conics*, the last one however not being completed by him. It seems that Apollonius took over the completion of these and added four more of his own. He defined the basic cone as one having an oblique axis and set on a circular base. This was called a scalene cone. He proved that all sections parallel to the base are also circular; and that there is another parallel set of circular sections, "sub-contrary" to these. In the latter part of this paper the author will invoke modern analytic geometry methods to determine the dihedral angle between these two sub-contrary sets of circular sections.

Apollonius derived the equivalent of the focus-directrix properties of the sections of this general cone, obtaining results equivalent to the modern analytic definitions of the conics. However, he made no reference to a directrix nor did he evince any knowledge thereof. Neither did he utilize a focus for the parabola. He showed that the conic has the same property with reference to any diameter as it has with reference to its axis (a special diameter). He expressed the fundamental property of each conic by equations between areas of associated rectangles. These relations were equivalent to the Cartesian equation of the conic. However, it remained for Pappas to state the modern locus definition for the conics. The latter first introduced the concept of eccentricity. Apollonius assigned the appropriate names to the sections, parabola (equivalent), ellipse (falls short) and hyperbola (exceeds). Straight lines and circles were then known as "plane loci", whereas conics were spoken of as "solid loci".

Archimedes derived further properties relating to chords, tangents, and diameters. His most unique contribution was that of Quadrature of the Parabola. This was an ingenious mechanical infinitesimal method, based upon principles of statics, considering areas as weights balanced about a fulcrum. He adhered to the then current definition of the three conics as arising from sections of three distinct kinds of circular cones (right-angled, obtuse-angled, and acute-angled) by planes drawn, in each case, at right angles to an element of the cone. But he was aware that an ellipse could be obtained by cutting a cone in any manner by a plane not parallel to the circular base, and also by cutting a cylinder.

The Greek geometers never knew that the properties of the conic sections could be inferred from the corresponding properties of the circle which forms the base of the cone. They treated the ellipse, parabola, and hyperbola as distinct curves. The unifying relations were not discovered until the 16th century when the principle of continuity elegantly established them – thanks to Kepler and Desargues.

The usual locus definitions of the conics, in terms of directrix and focus were beautifully developed in 1822 by G. P. Dandelin, a Belgian mathematician. By using his "Focal Spheres", often spoken of as the "Dandelin Spheres", which he inscribed tangent to the cone and to the section cut by the cutting plane, he established a complete analytic relation between cone and conic, with its foci. Seven years later, Morton extended this treatment by introducing directrices. However these basic properties were known to Pappas 2000 years before.

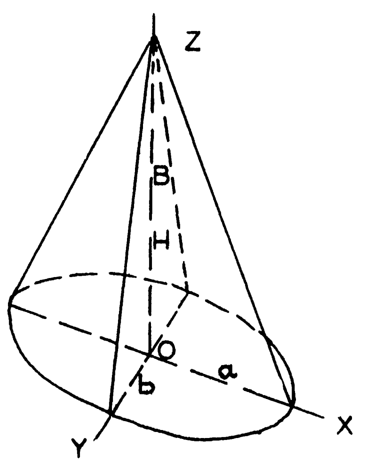
Desargues discovered the fundamental theorems of perspective, homology, poles and polars, and ideal elements. However, it remained for Pascal in his *Essai pour les coniques* (1640) to extend projective principles fully to the conic sections. At the age of 16 years he established his famous theorem on the inscribed hexagon, and from it deduced some 400 corollaries. It was not until 1806, that Brianchon established the dual of this theorem.

Philipe de la Hire (1640-1718) in his impressive *Sectiones Conicae* proved over three hundred projective theorems synthetically, and in a remarkable appendix showed that all the theorems of Apollonius are obtainable by the method of projection.

During the next 150 years, synthetic geometry again became dormant, but for a new reason: analytic geometry had blossomed forth, followed promptly by the calculus. However, a revival ensued toward the end of the 18th century. Among the mathematicians who had a distaste for analysis were geometers such as Monge, and his pupils, Poncelet, Brianchon, Chasles and Carnot. Other contributors were Lagrange, LaPlace, Gauss, Steiner, and Von Staudt. This surge of interest was maintained for over 100 years – in fact, through the first part of the 20th century. These developments were the last outstanding attempts to utilize the classical synthetic methods to the exclusion of the newer analytic techniques.

The nineteenth century was the productive period for the analytic treatment of conics. Through binary forms and the use of appropriate coordinate systems the entire analysis proceeded elegantly through the complete polar process.

A further short contribution by the author, presented below, will be accomplished by elementary methods. However, it illustrates the salient arguments of this paper, namely: first, the endless variety and novelty inherent in the cone itself; and secondly, the power and versatility of the analytic methods over the purely synthetic ones in investigating the properties of the cone. We shall seek now the angle between the two "subcontrary" sets of circular sections. We reverse our point of view here and shall attempt to cut *circular* cross-sections from elliptical cones, instead of elliptical ones from the conventional circular cone. Given a *right* cone with *elliptical* base, to cut it by planes making such an angle with the axis of the cone, as to produce *circular* sections. It is proposed to find the required angle θ in terms of α and β, where 2α is the greatest, and 2β is the minimum vertex angle at the vertex of the cone. Placing the cone's axis on the z-axis, and its base in the x-y plane, with $a > b$, (see figure)



then the equation of the cone will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{h-z}{h}\right)^2 .$$

A sphere $x^2 + y^2 + z^2 = r^2$, with undetermined radius, and center at the origin, intersects the cone in the locus:

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) - z^2\left(\frac{1}{h^2} + \frac{1}{r^2}\right) + \frac{2z}{h} = 0 .$$

This represents two planes if it factors into two real linear factors. This is possible if $r = a$. It then yields:

$$y^2 - z^2\ \frac{b^2}{h^2}\cdot\frac{a^2+h^2}{a^2-b^2} + \frac{2a^2b^2z}{h(a^2-b^2)} = 0\ .$$

Setting the coefficient of z^2 equal to m^2 gives:

$$(y+mz+\cdots)\cdot(y-mz+\cdots) = 0\ .$$

But $m = \tan\theta$; and since

$$\sin^2\alpha = \frac{a^2}{a^2+h^2}$$

and

$$\sin^2\beta = \frac{b^2}{b^2+h^2}\ ,$$

then by trigonometry it follows that

$$\sin\theta = \frac{\sin\beta}{\sin\alpha}\ .$$

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Note: The author wishes to thank Dr. F. D. Murnaghan for a suggestion that shortened the above solution.

District of Columbia Teachers College

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, 12, Texas.*

COMMENTS ON MATHEMATICS AND PHILATELY

Kurt R. Biermann

With great interest, I read the excellent essay by Maxey Brooke, "Mathematics and Philately," Volume 34, Number 1, September 1960, pp. 31-32. I would like to add to his list some memorial stamps which he did not mention. These include:

Newton, France, 1957, (18 Fr.)
Gauss, Germany, 1955, (10 Pf.)
Pascal, France, 1944, (1.20+2.80 Fr.)
Laplace, France, 1957, (30+9 Fr.)
Poincare, France, 1952, (18+5 Fr.)

Additional stamps have been dedicated to Huygens, Netherlands, 1928, (12½ c); deWitt, Netherlands, 1947, (7½ c); Monge, France, 1958, (18+5 Fr.); Lagrange, France, (8 Fr.); Euler, U. S. S. R., 1957, (40 K); Euler, Switzerland, 1957, (5+5 R); Euler, D. D. R., 1957, (10 Pf.); Vega, Yugoslavia, 1954, (50 D); Ostrograndski, U. S. S. R., 1951, (40 K); Kovalevskaya, U. S. S. R., 1951, (40 K); Krylov, U. S. S. R., 1955, (40 K); Ljapunov, U. S. S. R., 1957, (40 K); Riese, Germany, 1959, (10 Pf.); d'Alambert, France, (20+10 Fr.).

Finally, we can add to the list of Maxey Brooke another stamp dedicated to Lobatschefsky, U. S. S. R., 1946, (40 K). We can agree with Mr. Brooke that mathematicians appear rather infrequently on stamps, but the list is more complete than one would assume from his essay.

Berlin, Germany

BOOK REVIEWS

Modern textbooks and resource books are often improvements upon earlier books in these areas. However, it is often of great value to students to be able to refer to original sources or authoritative treatments of mathematical subjects. Reference to a textbook alone may create in the mind of a student only the formalistic aspects of mathematics. Whereas, reference to original sources and classical treatments as well as current texts can do much to broaden the student's understanding of the role of discovery and creation in mathematics as well as to develop an appreciation for mathematical style.

For this reason, I feel that the re-publication of some of the outstanding original treatises by such mathematicians as Boole, Forsyth, Cremona, and Eisenhart is an event of importance. An impressive list of such books that are classics in their fields has been published recently by Dover Publications, Inc. Of added importance to students in this day of high costs is the fact that these books appear in paperback form and are priced at only a fraction of the cost of hardback books.

The following list of books of this kind was received by the Mathematics Magazine from the publisher, Dover Publications, Inc., New York, all re-published in 1960.

Elements of Projective Geometry. By Luigi Cremona. xx+302 pp., $1.75.

Differential Geometry of Curves and Surfaces. By L. P. Eisenhart. xiv+474 pp., $2.75.

Advanced Euclidean Geometry. By R. A. Johnson. xiii+319 pp., $1.65.

Algebraic Equations. By Edgar Dehn. vii+208 pp., $1.45.

Transcendental and Algebraic Numbers. By A. O. Gelfond. vii+190 pp., $1.75.

The Theory of Equations. By W. S. Burnside and A. W. Panton. Vol. I, xiv+286 pp., Vol. II, xv+318 pp. Each $1.85.

A Treatise on the Calculus of Finite Differences. By George Boole. xii+336 pp., $1.85.

Theory of Maxima and Minima. By Harris Hancock. xiv+193 pp., $1.50.

Introduction to the Theory of Linear Differential Equations. By E. G. C. Poole. vii+202 pp., $1.65.
Calculus of Variations. By A. R. Forsyth. xxii+656 pp., $2.95.

Robert E. Horton

BOOKS RECEIVED FOR REVIEW

Arithmetic for the Modern Age. By Aaron Bakst. D. Van Nostrand Company, Inc., New York, 1960, vii+341 pp., $4.95.

Finite Difference Equations. By H. Levy and F. Lessman. The Macmillan Company, New York, 1961, vii+278 pp., $5.50.

Opportunities in Mathematics. By Harry M. Gehman. Vocational Guidance Manuals, Inc., New York, 1960, vi+74 pp., $1.65.

Great Ideas of Modern Mathematics: Their Nature and Use. By Jagit Singh. Dover Publications, Inc., New York, 1960, viii+312 pp., $1.55.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

446. *Proposed by David L. Silverman, Fort Meade, Maryland.*

Find the digital equivalents of the letters in the cryptaddition

```
 THREE
 EIGHT
  NINE
------
TWENTY
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(Dedicated to 6.0. 14522)

447. *Proposed by James W. Mellender, University of Wisconsin.*

Given two circles of radius x and y which are tangent externally and their circumcircle. Determine the radius of the circle tangent to the three given circles.

448. *Proposed by Brother U. Alfred, St. Mary's College, California.*

Determine an infinite series of terms such that each term of the series is a perfect square and the sum of the series at any point is a perfect square.

449. *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

For what positive integral x and for what $y = y(x)$ is the following inequality true?

$$(1 - 1/2x)(2x)^{2-2y} \leqq 1$$

450. *Proposed by Norman Anning, Sunnyvale, California.*

If the exponents m and n are positive integers, find the complete condition or conditions that $x^m + x^n + 1$ shall have a polynomial factor other than itself and 1.

451. *Proposed by B. L. Schwartz, Monterey, California.*

Let $S = \Sigma x_n$ be any conditionally convergent series of real terms. Let r be any real number. Prove there exists a conditionally convergent sub-series S' of S (obtained by deletion of terms without rearrangement) which converges to r.

452. *Proposed by H. Schwerdtfeger, McGill University, Montreal.*

Prove that all regular n by n matrices A with complex elements such that a certain complex vector x is eigen vector of A ($Ax = \alpha x$, with complex eigen value α) form a group $\mathbf{G}_x$ with respect to matrix multiplication.

SOLUTIONS

Late Solution

422. *J. L. Brown, Jr., Pennsylvania State University.*

Erratum

In Problem No. 437 (January 1961), P 174, the proposal should read: Prove or disprove the statement: The number of odd coefficients in the binomial expansion of $(a+b)^n$ is a power of 2, the exponent of 2 being the number of 1's appearing in the expression of n in the binary number system.

Euler's Phi-function

425. [November 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

I. *Solution by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.*

If $n+1$ and $n-1$ are prime, then n is both even and a multiple of 3, so that for some m, $n = 6m$, and we have:

$$\phi(n) = \phi(6)\,\phi(m) = 2\,\phi(m)\ ,$$

while

$$\phi(m) = m \prod_{p\,|\,m} \left(1 - \frac{1}{p}\right)\ ;$$

so

$$3\,\phi(n) = 6\,\phi(m) = 6m \prod_{p\,|\,m} \left(1 - \frac{1}{p}\right)\ ,$$

but $6m = n$ so,

$$3\,\phi(n) = n \prod_{p \mid m} (1 - \frac{1}{p}) ;$$

whence

$$3\,\phi(n) \leqq n ,$$

as required.

II. *Solution by L. Carlitz, Duke University.*

It is evidently necessary to assume $n > 4$. Since $n-1$ and $n+1$ are primes and $n > 4$ it follows that n is divisible by 3. Also n must be even so that n is divisible by 6. We shall now show that if

(1) $$n = 2^{\alpha}3^{\beta}m \quad (\alpha \geqq 1,\ \beta \geqq 1,\ (m, 6) = 1) ,$$

then

$$\phi(n) \leqq \frac{n}{3} .$$

Indeed from (1)

(2) $$\phi(n) = 2^{\alpha}3^{\beta-1}\phi(m) \leqq 2^{\alpha}3^{\beta-1}m = \frac{1}{3n} .$$

Remark: It is not difficult to show that

(3) $$\phi(n) = \frac{n}{3}$$

if and only if

(4) $$n = 2^{\alpha}3^{\beta} , \quad (\alpha \geqq 1,\ \beta \geqq 1) .$$

We have seen above that (4) implies (3). Now if (3) holds it is clear that n is divisible by 3. Put $n = 3^{\alpha}k$, where $\alpha \geqq 1$; then (3) becomes

$$2 \cdot 3^{\alpha}\phi(k) = n ,$$

so that n is even. Now put

$$n = 2^{\alpha}3^{\beta}m \quad (\alpha \geqq 1,\ \beta \geqq 1,\ (m, 6) = 1) .$$

Then if $m > 1$ it follows from (2) that

$$\phi(n) < \frac{n}{3} .$$

This completes the proof of the equivalence of (3) and (4).

Also solved by Brother Alfred, St. Mary's College, California; Leon Bankoff, Los Angeles, California; Maxey Brooke, Sweeney, Texas; B. A. Hausman, S. J., West Baden College, Indiana; Vern Hoggatt, San Jose State College; Sidney Kravitz, Dover, New Jersey; D. L. Silverman, Fort Meade, Maryland; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

A Set of Matrices

426. [November 1960] *Proposed by Dmitri Thoro, San Jose State College, California.*

Find the number N_p of non-singular matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose elements belong to the residue class modulo p, where p is a prime.

I. *Solution by Annie L. Laurer, New Haven, Connecticut.*

We partition the set of all matrices, whose elements belong to the residue class modulo p, into disjoint subsets S_4, S_3, S_2, S_1, S_0 consisting respectively of those matrices having exactly four, exactly three, exactly two, exactly one and exactly no elements equal to zero (where 0 (mod p) is meant throughout). We propose to count up the non-singular matrices in each set, and the sum will be N_p.

The sets S_4 and S_3 contain only singular matrices. For S_2 there are six possible arrangements of exactly two zeros, namely, the columns, the rows, and the diagonals. If the zeros are in one of the diagonals, the matrix is non-singular since the two remaining elements are different from zero, and hence, their product, and thus, the determinant is different from zero. The remaining matrices are singular. Thus, there are $2(p-1)^2$ non-singular matrices in S_2, since there are two diagonals and $(p-1)$ choices for each of the remaining two non-zero elements. All matrices in S_1 are non-singular since the product of the elements in one diagonal is zero and in the other is different from zero. There are $4(p-1)^3$ matrices in S_1 since there are four choices for where the zero will go and $(p-1)$ choices for each of the three remaining non-zero elements. In S_0 there are $(p-1)^4$ matrices, both singular and non-singular. If any three non-zero elements of a matrix of S_0 are given, then the fourth element is uniquely determined and different from zero for each singular matrix, since this amounts to finding the solution of a linear equation $ax-bc = 0$, where $a \neq 0$, $b \neq 0$, $c \neq 0$. Thus, there are $(p-1)^3$ singular matrices in S_0, and hence, the remaining $(p-1)^4 - (p-1)^3$ matrices are non-singular. Thus,

$$N_p = 0+0+2(p-1)^2+4(p-1)^3+(p-1)^4-(p-1)^3 = p(p+1)(p-1)^2 .$$

II. *Solution by D. W. Robinson, Brigham Young University.*

The problem is equivalent to finding the number of distinct ordered bases of the vector space of pairs of integers modulo p. Thus, since there are p^2 such vectors, and there are exactly p vectors linearly dependent on any given vector other than the zero vector, the required number

$$N_p = (p^2 - 1)(p^2 - p) .$$

This problem is a special case of the known result (see, for example, N. Jacobsen, *Lectures in Abstract Algebra*, Vol. II, p. 18) that the number of n by n unit matrices over a division ring of q elements is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) .$$

Also solved by F. D. Parker, University of Alaska, and the proposer. Two incomplete solutions were received.

A Cevian Relation

427. [November 1960] *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point in the plane of a triangle ABC through which cevians from B and C are drawn meeting sides CA and AB at points E and F reapectively. M is the midpoint of BC and line MP meets CA at N and AB at O. EF extended meets BC at G and a line through B parallel to AG meets CF at H. Show that HO is parallel to CA.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.

Let the points A, B, C, M and F be fixed and the geometrically interrelated points P, O, N, G, E, H be variable. Then from

$$O \overset{M}{\overline{\barwedge}} P \overset{B}{\overline{\barwedge}} E \overset{F}{\overline{\barwedge}} G \,\overline{\barwedge}\, AG \,\overline{\barwedge}\, BH \,\overline{\barwedge}\, H$$

we have $O \barwedge H$ of which F being the self corresponding element we deduce the perspectivity $O \,\overline{\barwedge}\, H$. Hence OH passes through a fixed point L. When O is at infinity on AB, H is also at infinity on CF, and hence L is at infinity. OH keeps then a fixed direction. But when $O \equiv B$, having $OH \equiv BH // AB$ the proof follows.

Also solved by the proposer.

Permuted Digits

428. [November 1960] *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

The number $N = 142{,}857$ has the property that $2N$, $3N$, $4N$, $5N$, and $6N$ are all permutations of N. Does there exist a number M such that $2M$, $3M$, $4M$, $5M$, $6M$, and $7M$ are all permutations of M?

I. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Since we get all permutations of M by $1M, 2M, \cdots, 7M$ the number M, if it exists, is a seven-digit number.

Let $M = abcdefg = Gg$ where $G = abcdef$ and let $1 \leqq p \leqq 7$ such that $p \cdot Gg = gG$. Then

$$p(10G + g) = 10^6 g + G$$

or

$$G = \frac{(10^6 - p)g}{(10p-1)} = N_p \cdot \frac{g}{Dp} .$$

Now

p	N_p	D_p	N_p/D_p	$(N_p/3)/D_p$
1	999,999	9	111,111	.
2	999,998	19	Irreducible	.
3	999,997	29	Irreducible	.
4	999,996	39 = 3.13	.	Irreducible
5	999,995	49 = 7.7	Irreducible	.
6	999,994	59	Irreducible	.
7	999,993	69 = 3.23	.	Irreducible

Since the coefficient N_p/D_p of g is not an integer except when $p = 1$, there is no solution for G other than $\overline{ggg,ggg}$. But $M = Gg = \overline{ggggggg}$ cannot be a solution.

Hence there is no solution to the problem.

II. *Comment by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.*

The number $M = 5882352941176470$ has the property that kM is a permutation of M for $k = 2, 3, \cdots, 16$. The number

$$L = 3448275862068965517241379310$$

has the property that kL is a permutation of L for $k = 2, 3, \cdots, 28$. (M consists of the digits in one cycle of the decimal expansion of $1/17$, and is 16 digits long, while L was similarly derived from $1/29$. I believe that it is correct that when p is prime and $1/p = Q$ has cycle length $p-1$, then kQ will be a permutation of Q for $k = 2, 3, \cdots, p-1$.)

A Well Known Summation

429. [November 1960] *Proposed by M. S. Krick, Albright College, Pennsylvania.*

Verify that

$$\sum_{k=1}^{n} 1/k\,[1 + (-1)^k \binom{n}{k}] = 0 .$$

Solution by F. D. Parker, University of Alaska.

Let

$$f(x) = \frac{1 - (1-x)^n}{x} .$$

Viewed as a geometric progression

$$f(x) = \sum_{k=0}^{n-1} (1-x)^k ,$$

and

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{n} \frac{1}{k} .$$

Viewed as a binomial expansion,

$$f(x) = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} x^{k-1}$$

and

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{n} \frac{1}{k}(-1)^{k-1}\binom{n}{k} .$$

Equating these expressions yields the desired result.

A number of solvers noted that this problem is equivalent to Problem No. 335 (March 1958) with solution appearing in November 1958, this Magazine. The problem also appears in *An Introduction to Probability Theory and Its Applications* by W. Feller.

Also solved by J. L. Brown, Jr., Pennsylvania State University; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Annie L. Laurer, New Haven, Connecticut; M. J. Pascual, Watervliet Arsenal, New York; L. A. Ringenberg, Eastern Illinois University; C. D. Sutherland, Watervliet Arsenal, New York; and the proposer.

A Harmonic Mean

430. [November 1960]. *Proposed by Leon Bankoff, Los Angeles, California.*

At a point P on the latus rectum of a parabola, a perpendicular to the latus rectum is erected, cutting the curve at Q. Show that PQ is half the harmonic mean of AP and PB.

Solution by M. J. Pascual, Watervliet Arsenal, New York.

There is no loss of generality in letting the parabola have the rectangular equation:

$$y^2 = 4px \quad \text{with } p > 0$$

in which the segment with end-points $A\,(p,\ 2p)$ and $B\,(p,\ -2p)$ is the latus rectum, and $F(p, 0)$ is the focus. If P lies between A and F, we easily find

$$PQ = p - \frac{y^2}{4p}, \quad AP = 2p - y, \quad PB = 2p + y$$

so that the harmonic mean HM of AP and PB is given by

$$HM = \frac{2}{\dfrac{1}{2p-y} + \dfrac{1}{2p+y}} = 2\left(p - \frac{y^2}{4p}\right).$$

Hence $\frac{1}{2}HM = PQ$. A similar argument holds if P lies between F and B. For P lying on AB extended and not between A and B we find that (using signed distances along AB)

$$PQ = \frac{y^2}{4p} - p, \quad AP = -y + 2p, \quad PB = y + 2p$$

so that

$$HM = \frac{2}{\dfrac{1}{-y+2p} + \dfrac{1}{y+2p}} = 2\left(\frac{y^2}{4p} - p\right).$$

Also solved by Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Electric Products, Inc.; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; James W. Mellender, University of Wisconsin; F. D. Parker, University of Alaska; Lawrence A. Ringenberg, Eastern Illinois University; Sister M. Stephanie, Georgian Court College, New Jersey; Harvey Walden, Rensselaer Polytechnic Institute; Hazel L. Wilson, Jacksonville University, Florida; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

Random Walk

431. [November 1960] *Proposed by William Squire, Southwest Research Institute, San Antonio, Texas.*

Given a rectangular array of numbers

$$\begin{matrix} 1 & 2 & 3 & 4 & \cdots & N \\ 2 & 3 & 4 & 5 & \cdots & N+1 \\ 3 & 4 & 5 & 6 & \cdots & N+2 \\ \vdots & & & & & \vdots \\ M & & & & & M+N-1 \end{matrix}.$$

How many paths are there going in correct numerical order from 1 to $M+N-1$?

Solution by Leo von Gottfried, Lawrence Radiation Laboratory, University of California.

Problem 431 may be treated as a restricted random walk. There are two choices available at each point in the array, except for the terminal, or high order corner, there being clearly only one choice at the two points adjacent to the terminal point. Then each point may be characterized by the number of ways it can be reached. Each point in the upper and left hand boundaries is accessible from only one point, its antecedent, and hence can be reached in only one way. All other points are accessible from two points and the number of ways such a point can be reached is the sum of the ways the two antecedent points can be reached. The array of weights may be written down immediately, and the beginning of such an array is shown in the Figure. From the nature of the process, the weights (or numbers of paths) will be binomial coefficients.

Figure 1
Weights of Points in Array

0	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35

Clearly, the weight of a point at (m, n) will involve $(m-1)$ and $(n-1)$, rather than m and n. Further the symmetry of the problem requires that the binomial coefficient involve the sum of $(m-1)$ and $(n-1)$. Hence the answer is

$$N(n, m) = \frac{(n-1+m-1)!}{(n-1)!(m-1)!} = \binom{n+m-2}{m-1}.$$

It may be easily verified that the solution satisfies the recursion relation, pointed out by Dr. Walter Aron of this laboratory:

$$N(n+1, m) = \sum_{k=1}^{m} N(n, m), \quad n \geqq m .$$

Also solved by Brother U. Alfred, St. Mary's College, California; Maxey Brooke, Sweeny, Texas; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Edward T. Frankel, U. S. Department of Health, Education, and Welfare; Michael J. Z. Kascei, Jr., St. Joseph's College, Pennsylvania; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; F. D. Parker, University of Alaska; Michael J. Pascual, Watervliet Arsenal, New York; Lawrence A. Ringenberg, Eastern Illinois University; D. L. Silverman, Fort Meade, Maryland; Harvey Walden, Rensselaer Polytechnic Institute; and the proposer.

Comment on Problem 415

415. [May 1960, January 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos (p)x \sin (n-p)x = 2^{n-1} \sin nx \ .$$

Comment by Louis Brand, University of Houston.

In the problem of a trigonometric sum a much simpler solution is as follows: Call the sum S and make the index change $p = n-q$; adding the two sums now gives

$$2S = \sum_{p=0}^{n} \binom{n}{p} \sin nx = 2^{n} \sin nx \ .$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 281. Evaluate the radius of the inner tritangent circle to excircles of a triangle [*Submitted by Huseyin Demir*].

Q 282. Prove that

$$\frac{\sin n\theta}{\sin \theta} \leqq 2^{n-1}\ , \quad \sin \theta \neq 0 \ .$$

[*Submitted by Barney Bissinger*].

Q 283. From Bauer and Brooke, *Plane and Spherical Trigonometry*, D. C. Heath and Co., 1917, p. 108: "Prove that the areas of an equilateral triangle and of a regular hexagon, of equal perimeters, are to each other as 2 : 3." [*Submitted by C. W. Trigg*].

Q 284. What is the locus of points whose projections on the sides of a triangle are collinear? [*Submitted by Huseyin Demir*].

Q 285. In what number system does the following property hold: If a number is even, then the number formed by any permutation of its digits is also even. [*Submitted by Brother U. Alfred*].

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 44. If

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

with r_1 and r_2 given, determine a simple construction for R. [*Submitted by Norman Anning*].

T 45. Determine the equation of the conic passing through the five points $(-3, -2)$, $(-2, 3)$, $(1, 1)$, $(-1, 1)$, $(4, -1)$. [*Submitted by M. S. Klamkin*].

(Answers to Quickies and Trickies are below.)

Answers

A 280. See **Q 280** on page 244, Vol. 34, No. 4, March, 1961, this magazine. The Fibonacci series is of the form $a, b, a+b, a+2b, \cdots$. With $a = 1$, $b = 13$ and the thirteenth term, $89a + 144b = 1961$, we obtain the series $1, 13, 14, 27, \cdots, 1961, \cdots$

The following submitted solutions to the proposer in the order listed. The first two names were the winners of the gift packages of Wisconsin cheese. *David Friedman, California Institute of Technology; Monte Dernham, San Mateo, California; Wayne H. Jones, Pacific Palisades, California; Benjamin L. Schwartz, Pebble Beach, California; Michael L. Cantor, New York, New York; Edwin Comfort, Ripon College, Wisconsin; Sister Mary Constantia, Ward High School, Kansas City, Kansas; M. E. White, Stevens Institute of Technology; H. C. McKenzie, South Dakota State College; Edgar Karst, Provo, Utah; Dave Druten, Kansas City, Kansas; Herbert R. Leifer, Pittsburgh, Pennsylvania; C. F. Pinzka, University of Cincinnati; Darryl Kuhns, Troy, Ohio; Marlow Sholander, Shaker Heights, Ohio; Paul Stygar, New Haven, Connecticut; V. E. Hoggatt, Santa Clara, California; Julian H. Braun, Chicago, Illinois; Melvin Hochster, Harvard College; Joseph D. E. Konhauser, State College, Pennsylvania; E. P. Miles, Florida State University; William R. Ransom, Reading, Massachusetts; C. L. Ackerman, The Pennsylvania State University; Sidney Kravitz, Dover, New Jersey; Harry J. Saal, Columbia College, New York; James A. Ferris, Broderick, California; Herta T. Freitag, Hollins, Virginia; Merrill Barnebey, Grand Forks, North Dakota; Nello Allegrezza, Milford High School, Milford, Massachusetts; Brother Louis Zirbel, Marist High School, Bayonne, New Jersey; and Donald K. Bissonnette, Florida State University.*

(Answers to Quickies and Solutions to Trickies appearing on pages 308-309).

A281. This circle, being the nine-point circle of the triangle, has radius $\frac{1}{2}R$.

A282. The result follows at once from the identity

$$\sin n\theta = 2^{n-1}\sin\theta\sin\left(\theta+\frac{2\pi}{n}\right)\cdots\sin\left[\theta+\frac{(n-1)\pi}{n}\right].$$

A283. The sides of the triangle and hexagon are in the ratio 2 : 1. Hence, the triangle can be dissected into 4 equilateral triangles and the hexagon into six equilateral triangles, all congruent. So the areas are in the ratio of 2 : 3.

A284. If the points are restricted to lie on the plane of the triangle, the locus is the circumcircle of the triangle. Since no such restriction is made, the locus is the right cylinder having this circumcircle as section.

A285. Any number system with an odd base will do. For $b \equiv 1 \pmod 2$. Let

$$N = a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n .$$

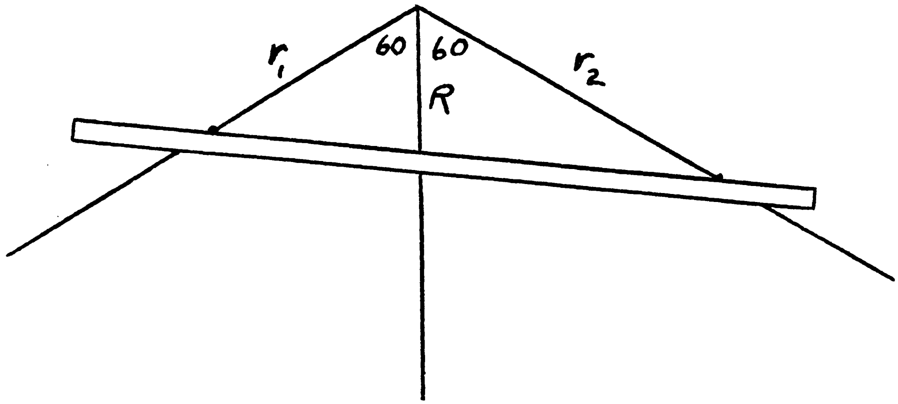
Then

$$N \equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \pmod 2$$

and this will continue even or odd no matter how the digits are permuted.

Solutions

S44. To construct R, lay out r_1 and r_2, apply a straight edge and pick off R on the angle bisector.



S45. Since (4, −1), (1, 1), and (−2, 3) are collinear, the conic degenerates into the two straight lines

$$(2x+3y-5)(3x-2y+5) = 0 .$$

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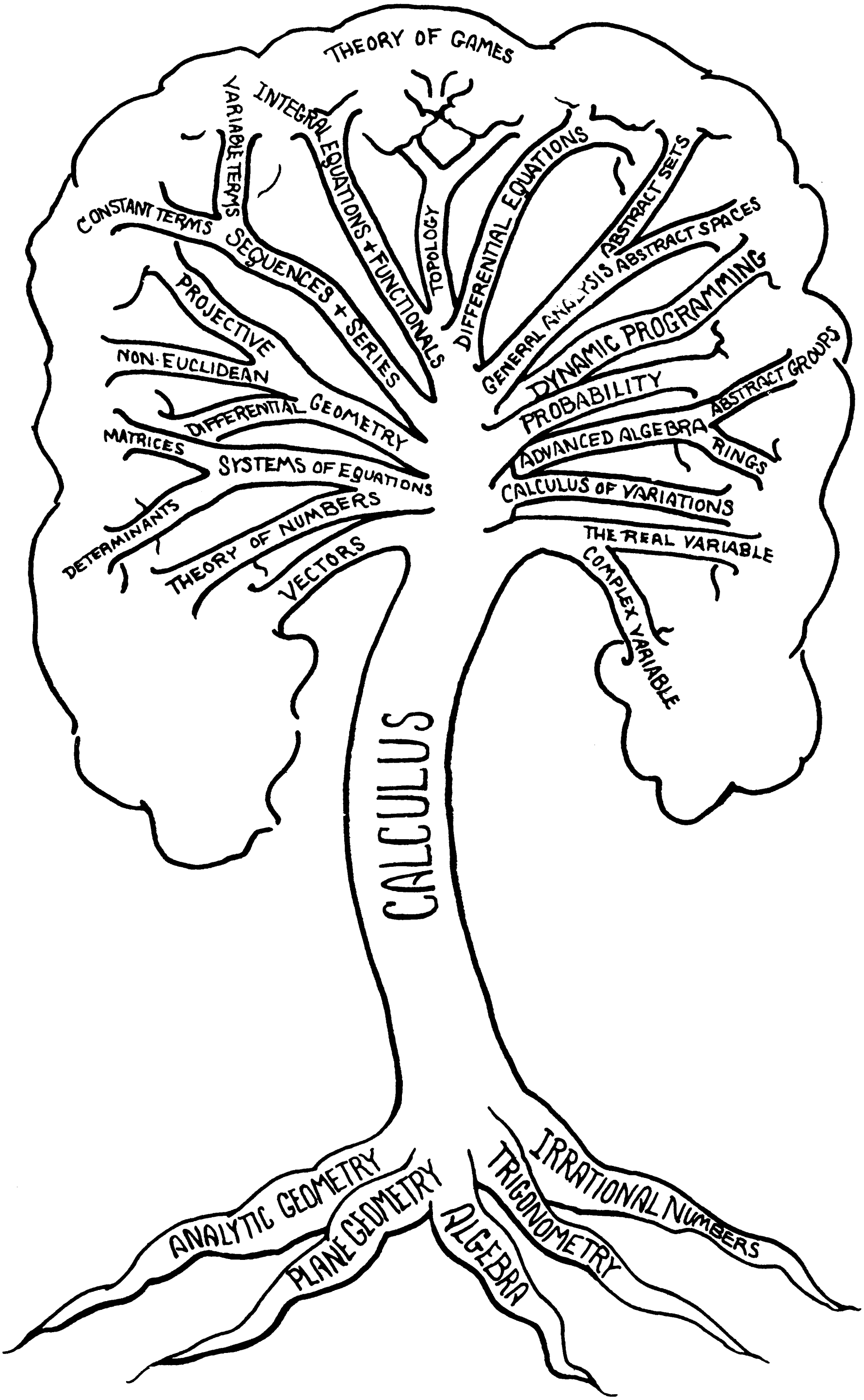

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